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MISO Intuitionistic Fuzzy Inference System

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Abstract

Fuzzy Inference Systems (FIS) are widely used for decision-making through imprecise relation defined on qualitative imprecise inputs and outputs based on expert's knowledge via fuzzy numbers. Even the fuzzy relation is based on the expert's knowledge, it is not accounted with the expert's lack of confidence / hesitancy, if any, involved in the relation between qualitative imprecise inputs and outputs because of their imprecise / fuzzy in nature. This research introduces an enhanced Intuitionistic Fuzzy Inference System (IFIS) to overcome the limitations of conventional Fuzzy Inference Systems (FIS) in handling imprecise, incomplete, and uncertain expert's data by considering expert's hesitancy / lack of knowledge in domain. IFIS extends traditional fuzzy models by incorporating intuitionistic fuzzification, intuitionistic IF-THEN implications, and intuitionistic defuzzification, all of which account for expert's hesitation / lack of confidence if any due to lack of knowledge through an α - level hesitancy parameter. A Multi-Input Single-Output (MISO) intuitionistic fuzzy system is developed as a generalization of the Single-Input Single-Output (SISO) model. To demonstrate the utility of this approach, the study applies a trapezoidal intuitionistic fuzzy inference system (TIFIS) to model COVID-19 risk, assuming expert data may exhibit varying degrees of confidence. This novel framework significantly enhances decision-making processes in complex, uncertain environments, offering a robust alternative to existing FIS models.

Keywords: MISO Intuitionistic fuzzy inference mechanisms, Intuitionistic defuzzification, Trapezoidal intuitionistic fuzzification with hesitancy, COVID-19 intuitionistic fuzzy model.

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1|Introduction

Approximate reasoning is very crucial for deriving actionable insights from ambiguous inputs by establishing relationships between qualititative imprecise inputs and outputs in real-world scenarios. Zadeh's foundational works [56], [57], [58], [59], [60] on this subject laid the groundwork for Fuzzy Inference Systems (FIS), which typically function through three primary stages: fuzzification, inference mechanism, and defuzzification based on the fuzzy numbers which model expert's knowledge on the domain of imprecise inputs and outputs. These systems have been widely applied across various domains, including artificial intelligence, decision analysis, and expert systems [3], [4], [11], [14], [36], [37], [40], [47], [49] proving their utility in complex decision-making environments.

1.1|Limitations of Existing Fuzzy Inference Systems

Despite their widespread adoption, traditional FIS [3], [4], [12], [13], [14], [28], [30], [31], [34], [36], [37], [40] face significant limitations, particularly when addressing incomplete or vague data due to the hesitancy / lack of knowledge on the domain of imprecise inputs and outputs. The inherent structure of fuzzy sets is based solely on membership function with the assumption that non-membership values are 1 minus membership values, which may not capture completely the nuances of uncertainty present in real-world situations. When expert's inputs are uncertain due to lack of confidence / incomplete on knowledge domain, conventional fuzzy logic systems can yield misleading or overly simplistic conclusions. This limitation emphasizes the urgent need for more sophisticated approaches that can effectively address the complexities of uncertainty and enhance the reliability of decision-making processes as a generalisation of FIS to apply in both scenarios with or without lack of confidence / hesitancy in the knowledge domain.

2|Preliminary Concepts

In order to make this paper self-contained, we briefly discuss a few results and findings that will be used later on in this work. [6] Let $U \neq \emptyset$. An Intuitionistic fuzzy set (IFS) $A = \{\langle u, \mu_A(u), \nu_A(u)\rangle \mid u \in U\}$, where $\mu_A : U \longrightarrow [0,1]$ and $\nu_A(u) : U \longrightarrow [0,1]$, $u \in U$ with the constraint $0 \leq \mu_A(u) + \nu_A(u) \leq 1, \ \forall\ u \in U$ and $\mu_A(u), \nu_A(u) \in [0,1]$ denote the degree of membership and non-membership of u to be in A respectively. For each intuitionistic fuzzy set A in U, $\pi_A(u) = 1 - \mu_A(u) - \nu_A(u)$ is labeled as hesitancy or uncertainty degree of u to be in A. The IFS A is denoted as (μ_A, ν_A).

[24] The fuzzy T - norm is a function $\hat{T} : [0,1]\times[0,1] \longrightarrow [0,1]$ such that $\hat{T}(t,1) = t, \ \forall\ t \in [0,1]$ with monotonically increasing, associative and commutative properties. The fuzzy T- conorm is a function $\check{S} : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $\check{S}(t,0) = t, \ \forall\ t \in [0,1]$ with monotonically increasing, associative and commutative properties. For example, $\hat{T}(u,v) = u \vee v = \max\{u,v\}$ and $\check{S}(u,v) = u \wedge v = \min\{u,v\}$, where $u, v \in [0,1]$.

[18] Let A and B be two IFSs in universe U. The union of A and B is defined as, $A \cup B = \{\langle u, \max\{\mu_A(u), \mu_B(u)\}, \min\{\nu_A(u), \nu_B(u)\}\rangle \mid u \in U\}$. The intersection of A and B is defined as, $A \cap B = \{\langle u, \min\{\mu_A(u), \mu_B(u)\}, \max\{\nu_A(u), \nu_B(u)\}\rangle \mid u \in U\}$. [9] An intuitionistic fuzzy number $A = (\mu_A, \nu_A)$ in the set of real numbers $\mathbb{R}$ is defined as

$$\mu_A(u) = \begin{cases} l_A(u) & \text{if} \quad w_1 \leq u \leq a \\ 1 & \text{if} \quad a \leq u \leq b \\ r_A(u) & \text{if} \quad b \leq u \leq w_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_A(u) = \begin{cases} l'_A(u) & \text{if} \quad u_1 \leq u \leq c \\ 0 & \text{if} \quad c \leq u \leq d \\ r'_A(u) & \text{if} \quad d \leq u \leq u_2 \\ 1, & \text{otherwise} \end{cases}$$

where $0 \leq \mu_A(u) + \nu_A(u) \leq 1$ and $w_1, a, b, w_2, u_1, c, d, u_2 \in \mathbb{R}$ such that $u_1 \leq w_1, c \leq a \leq b \leq d, w_2 \leq u_2$, and four functions $l_A, r_A, l'_A, r'_A : \mathbb{R} \to [0,1]$ are the legs of membership function μ_A and non-membership function ν_A. The functions l_A and r'_A are non-decreasing continuous functions and the functions l'_A and r_A are non-increasing continuous functions. An intuitionistic fuzzy number is denoted as $[(w_1, a, b, w_2), (u_1, c, d, u_2)]$, where $(u_1, c, d, u_2) \leq (w_1, a, b, w_2)^c$. [52] A trapezoidal intuitionistic fuzzy number $A = (\mu_A, \nu_A)$ with parameters

$u_1 \leq w_1, c \leq a \leq b \leq d, w_2 \leq u_2$ in the set of real numbers $\mathbb{R}$ is defined as

$$\mu_A(u) = \begin{cases} \frac{u-w_1}{a-w_1} & \text{if} \quad w_1 \leq u \leq a \\ 1 & \text{if} \quad a \leq u \leq b \\ \frac{w_2-u}{w_2-b} & \text{if} \quad b \leq u \leq w_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_A(u) = \begin{cases} \frac{c-u}{c-u_1} & \text{if} \quad u_1 \leq u \leq c \\ 0 & \text{if} \quad c \leq u \leq d \\ \frac{u-d}{u_2-d} & \text{if} \quad d \leq u \leq u_2 \\ 1, & \text{otherwise} \end{cases}$$

and it is denoted as $A = [(w_1, a, b, w_2), (u_1, c, d, u_2)]$. [8] Let $U \neq \emptyset$, $V \neq \emptyset$ and $A = (\mu_A, \nu_A),\ B = (\mu_B, \nu_B)$ be two IFSs of U and V respectively. The Cartesian product $A \times B$ of $U \times V$ defined as $A \times B = (\mu_{A\times B}, \nu_{A\times B})$ where $\mu_{A\times B}(u, v) = \min\{\mu_A(u), \mu_B(v)\}$ and $\nu_{A\times B}(u, v) = \max\{\nu_A(u), \nu_B(v)\}, \forall u \in U, v \in V$. [10] Let $U \neq \emptyset$, $V \neq \emptyset$ and $A = (\mu_A, \nu_A),\ B = (\mu_B, \nu_B)$ be two IFSs of U and V respectively. An IFS R of $U \times V$ is called intuitionistic fuzzy relation (IFR) from A to B if

$$\begin{aligned} &(i)\ \mu_R(u, v) \leq \mu_{A\times B}(u, v), \\ &(ii)\ \nu_R(u, v) \geq \nu_{A\times B}(u, v), \\ &(iii)\ \mu_R(u, v) + \nu_R(u, v) \leq 1. \end{aligned}$$

The set of all intuitionistic fuzzy relations from A to B is denoted as $IFR(A, B)$. [8] Let $A = (\mu_A, \nu_A),\ B = (\mu_B, \nu_B)$ be two IFSs of U and V respectively and R be an IFR from A to B. Let $A' = (\mu_{A'}, \nu_{A'})$ be IFS on U. The composition $\circ$ for R and A' defined as $A' \circ R = (\mu_{A'\circ R}, \nu_{A'\circ R})$ where, $\mu_{A'\circ R}(v) = \sup\limits_u \min\{\mu_{A'}(u), \mu_R(u, v)\}$ and $\nu_{A'\circ R}(v) = \inf\limits_u \max\{\nu_{A'}(u), \nu_R(u, v)\}$. [38] Let $U \neq \emptyset$. An intuitionistic fuzzy singleton $A' = (\mu_{A'_v}, \nu_{A'_v})$ defined on v is defined as

$$A' = \left(\mu_{A'_v}(z) = \begin{cases} \alpha & \text{if} \quad z = v \\ 0, & \text{otherwise} \end{cases}, \quad \nu_{A'_v}(z) = \begin{cases} \beta & \text{if} \quad z = v \\ 1, & \text{otherwise} \end{cases} \right)$$

such that $\alpha \in (0, 1], \beta \in [0, 1)$ with $0 < \alpha + \beta \leq 1$.

3|LOEX Option

An intuitionistic fuzzy IF-THEN rule is frequently referred to as an intuitionistic fuzzy implications which takes the form of

$$\text{IF } u \text{ is } A, \text{ THEN } w \text{ is } C$$

where A and C are linguistic values defined on the universes of discourse U and W respectively which are modeled by IFSs with respect to the universe of discourse U and universe of discourse W, respectively. In this implication, $u\ is\ A$ is frequently referred to as the premise, whereas $w\ is\ C$ is the conclusion. Alternatively, the IF-THEN implication can be abridged as $A \to C$. Furthermore, this $A \to C$ will be a fuzzy subset $A \times C$ of $U \times W$. Following this, $I(A, C)$ is defined as $I(A, C) := A \times C = (\mu_{A\times C}, \nu_{A\times C})$. For clear comprehension, some valid examples of IFIs are discussed below, along with the notations given above.

Mamdani IFI : The Mamdani IFI A ->$[I_M]$ C is defined as $I_M(A, C) = (\mu_{I_M(A,C)}, \nu_{I_M(A,C)})$, where

$$\begin{aligned} \mu_{I_M(A,C)}(u, w) &= \mu_A(u) \wedge \mu_C(w) = \min\{\mu_A(u), \mu_C(w)\}, \\ \nu_{I_M(A,C)}(u, w) &= \nu_A(u) \vee \nu_C(w) = \max\{\nu_A(u), \nu_C(w)\}. \end{aligned} \tag{1}$$

Larsen IFI : The Larsen IFI A ->$[I_L]$ C is defined as $I_L(A, C) = (\mu_{I_L(A,C)}, \nu_{I_L(A,C)})$, where

$$\begin{aligned} \mu_{I_L(A,C)} &= \mu_A(u) \cdot \mu_C(w), \\ \nu_{I_L(A,C)} &= \nu_A(u) + \nu_C(w) - \nu_A(u) \cdot \nu_C(w). \end{aligned} \tag{2}$$

Bounded IFI : The Bounded IFI A ->$[I_B]$ C is defined as $I_B(A, C) = (\mu_{I_B(A,C)}, \nu_{I_B(A,C)})$, where

$$\begin{aligned} \mu_{I_B(A,C)} &= \max\{0, \mu_A(u) + \mu_C(w) - 1\}, \\ \nu_{I_B(A,C)} &= \min\{1, \nu_A(u) + \nu_C(w)\}. \end{aligned} \tag{3}$$

Drastic IFI : The Drastic IFI A ->$[I_D]$ C is defined as $I_D(A,C) = (\mu_{I_D(A,C)}, \nu_{I_D(A,C)})$, where

$$\mu_{I_D(A,C)} = \begin{cases} \mu_A(u) & \text{if } \mu_C(w) = 1 \\ \mu_C(w) & \text{if } \mu_A(u) = 1 \\ 0 & \text{if } \mu_A(u), \mu_C(w) < 1 \end{cases} \quad \text{and} \quad \nu_{I_D(A,C)} = \begin{cases} \nu_A(u) & \text{if } \nu_C(w) = 0 \\ \nu_C(w) & \text{if } \nu_A(u) = 0 \\ 1 & \text{if } \nu_A(u), \nu_C(w) > 0 \end{cases}. \tag{4}$$

In an Intuitionistic Fuzzy Inference System (IFIS), the conclusion of one Intuitionistic Fuzzy Implication (IFI) or Intuitionistic Fuzzy Relation (IFR) is not propagated to the premise of another, which simplifies the inference process in comparison to traditional expert systems. This makes the intuitionistic inference mechanism used in IFIS more accessible and computationally efficient. Typically, the rule base in an IFIS is derived from expert knowledge, and its structure follows a multi-input and multi-output (MIMO) system. In this paper, we focus on a specific case, where the system operates with multiple inputs and a single output (MISO).

The intuitionistic fuzzy rule base I_{MISO} for a system with n inputs and a single output, comprising m rules, can be represented as:

$$I = \{I^1_{MISO}, I^2_{MISO}, \ldots, I^m_{MISO}\},$$

where each rule I^j_{MISO} takes the form:

$$\text{IF } x^*_1 \text{ is } A^*_{1j} \text{ and } x^*_2 \text{ is } A^*_{2j} \text{ and} \ldots \text{and } x^*_n \text{ is } A^*_{nj}, \text{ THEN } z^* \text{ is } C^*_j.$$

Here, the premise of I^j_{MISO} induces an intuitionistic fuzzy set (IFS) $A^*_{1j} \times A^*_{2j} \times \cdots \times A^*_{nj}$ in the product space $U_1 \times U_2 \times \cdots \times U_n$, and the conclusion is the IFS C^*_j in W, where U_j represents the universe of discourse of input variable x^*_j and W represents the universe of discourse for output variable z^*.

The j^{th} implication (rule) I^j_{MISO} can be expressed as:

$$I^j_{MISO} : A^*_{1j} \times A^*_{2j} \times \cdots \times A^*_{nj} \longrightarrow C^*_j.$$

Consequently, the entire rule base I can be written as:

$$I = \bigcup_{j=1}^{m} \left[(A^*_{1j} \times A^*_{2j} \times \cdots \times A^*_{nj}) \longrightarrow C^*_j \right].$$

Now, consider the generic form of a MISO IFI with two inputs $A^*_j(\mu_{A^*_j}, \nu_{A^*_j})$, $B^*_j(\mu_{B^*_j}, \nu_{B^*_j})$, and a single output $C^*_j(\mu_{C^*_j}, \nu_{C^*_j})$. The framework for two inputs can be extended to "n" inputs and a single output as follows:

$$\begin{aligned} I_1 &: \text{if } x^* \text{ is } A^*_1 \text{ and } y^* \text{ is } B^*_1, \text{ then } w^* \text{ is } C^*_1, \\ I_2 &: \text{if } x^* \text{ is } A^*_2 \text{ and } y^* \text{ is } B^*_2, \text{ then } w^* \text{ is } C^*_2, \\ &\ldots \\ I_m &: \text{if } x^* \text{ is } A^*_m \text{ and } y^* \text{ is } B^*_m, \text{ then } w^* \text{ is } C^*_m. \end{aligned}$$

Each IFI $I_j = (\mu_{I_j}, \nu_{I_j})$ is described as:

$$\begin{aligned} \mu_{I_j} &= \mu_{(A^*_j \text{ and } B^*_j \longrightarrow C^*_j)}(u, v, w) \\ &= [\mu_{A^*_j}(u) \text{ and } \mu_{B^*_j}(v)] \longrightarrow \mu_{C^*_j}(w), \\ \nu_{I_j} &= \nu_{(A^*_j \text{ and } B^*_j \longrightarrow C^*_j)}(u, v, w) \\ &= [\nu_{A^*_j}(u) \text{ and } \nu_{B^*_j}(v)] \longrightarrow \nu_{C^*_j}(w), \end{aligned}$$

where "A^*_j and B^*_j" is the IFS $A^*_j \times B^*_j$ in $U \times V$, and $I_j = (A^*_j(u) \text{ and } B^*_j(v)) \longrightarrow \mu_{C^*_j}(w)$ is the IFI in $U \times V \times W$.

We now demonstrate the output C^*_j for the two inputs A' and B' using generalized modus ponens from the j^{th} implication, given by the following compositional equation:

$$C'_j(\mu_{C'_j}, \nu_{C'_j}) = \left(\sup_{u,v} \min \left(\mu_{A'(u)}, \mu_{B'(v)}, \mu_{I_j}(u, v, w) \right), \inf_{u,v} \max \left(\nu_{A'(u)}, \nu_{B'(v)}, \nu_{I_j}(u, v, w) \right) \right). \tag{5}$$

Finally, the output C' for the two inputs A' and B' from all m implications I_j can be written as:

$$C'(\mu_{C'}, \nu_{C'}) = (A', B') \circ \bigcup_{j=1}^{m} I_j,$$

where $\circ$ represents the intuitionistic fuzzy sup-min compositional operator, and the resulting expression is:

$$C'(\mu_{C'}, \nu_{C'}) = \left(\sup_{u,v} \min \left(\mu_{A'(u)}, \mu_{B'(v)}, \max_{j=1,\ldots,m} \mu_{I_j}(u,v,w) \right), \inf_{u,v} \max \left(\nu_{A'(u)}, \nu_{B'(v)}, \min_{j=1,\ldots,m} \nu_{I_j}(u,v,w) \right) \right). \quad (6)$$

Here, A' and B' are intuitionistic fuzzy inputs, I_j is the j^{th} intuitionistic fuzzy implication (relation), and $\circ$ represents the sup-min composition.

Let $A^{'} = (\mu_{A'}, \nu_{A'})$ and $B^{'} = (\mu_{B'}, \nu_{B'})$ be two intuitionistic fuzzy inputs and let $I_j = (\mu_{I_j}, \nu_{I_j})$ be intuitionistic fuzzy implications for $j = 1, 2, \ldots, m$. Then the non-membership of the output $C' = (\mu_{C'}, \nu_{C'})$ is given by:

$$\nu_{C'} = (\nu_{A'}, \nu_{B'}) \circ \nu_{\bigcup_{j=1}^{m} I_j} = \bigcap_{j=1}^{m} (\nu_{A'}, \nu_{B'}) \circ \nu_{I_j}.$$

Proof: We start by expressing the non-membership function for the output C' in terms of the inputs and implications:

$$\begin{aligned}
\nu_{C'}(w) &= (\nu_{A'}, \nu_{B'}) \circ \nu_{\bigcup_{j=1}^{m} I_j}(w) \\
&= \inf_{u,v} \max \left\{ \nu_{A'}(u), \nu_{B'}(v), \min \left[\nu_{I_1}(u,v,w), \nu_{I_2}(u,v,w), \ldots, \nu_{I_m}(u,v,w) \right] \right\} \\
&= \min \left\{ \inf_{u,v} \max \left(\nu_{A'(u)}, \nu_{B'(v)}, \nu_{I_1}(u,v,w) \right), \ldots, \inf_{u,v} \max \left(\nu_{A'(u)}, \nu_{B'(v)}, \nu_{I_m}(u,v,w) \right) \right\} \\
&= \min \left\{ [(\nu_{A'}, \nu_{B'}) \circ \nu_{I_1}](w), \ldots, [(\nu_{A'}, \nu_{B'}) \circ \nu_{I_m}](w) \right\} \\
&= \bigcap_{j=1}^{m} [(\nu_{A'}, \nu_{B'}) \circ \nu_{I_j}](w).
\end{aligned}$$

Thus, we have shown that:

$$\nu_{C'} = \bigcap_{j=1}^{m} \nu_{C'_j}, \quad \text{where } C'_j = (\mu_{C'_j}, \nu_{C'_j}) \text{ is the output for each rule } I_j.$$

$\square$

For the output $C' = (\mu_{C'}, \nu_{C'})$, it holds that:

$$0 \leq \mu_{C'}(w) + \nu_{C'}(w) \leq 1.$$

Proof: From the definitions of membership and non-membership functions, we know that:

$$\mu_{C'}(w) = \sup_{u,v} \min \left\{ \mu_{A'}(u), \mu_{B'}(v), \max_{j=1}^{m} \mu_{I_j}(u,v,w) \right\}$$

and

$$\nu_{C'}(w) = \inf_{u,v} \max \left\{ \nu_{A'}(u), \nu_{B'}(v), \min_{j=1}^{m} \nu_{I_j}(u,v,w) \right\}.$$

We need to show that:

$$0 \leq \mu_{C'}(w) + \nu_{C'}(w) \leq 1.$$

Since:

$$1 = \sup_{u,v} \min \left\{ \mu_{A'}(u), \mu_{B'}(v), \max_{j=1}^{m} \mu_{I_j}(u,v,w) \right\} + \inf_{u,v} \left(1 - \min \left\{ \mu_{A'}(u), \mu_{B'}(v), \max_{j=1}^{m} \mu_{I_j}(u,v,w) \right\} \right),$$

and noting that $\nu_{A'} \leq 1 - \mu_{A'}$, $\nu_{B'} \leq 1 - \mu_{B'}$, and $\nu_{I_j} \leq 1 - \mu_{I_j}$, we obtain: $0 \leq \mu_{C'}(w) + \nu_{C'}(w) \leq 1$. $\square$

Let $A' = (\mu_{A'}, \nu_{A'})$ and $B' = (\mu_{B'}, \nu_{B'})$ be two intuitionistic fuzzy inputs, and define the following intuitionistic fuzzy implications (IFIs):

$$\begin{aligned}
I_j(\mu_{I_j}, \nu_{I_j}) &: \quad \text{If } u \text{ is } A_j(\mu_{A_j}, \nu_{A_j}) \text{ and } v \text{ is } B_j(\mu_{B_j}, \nu_{B_j}), \text{ then } w \text{ is } C_j(\mu_{C_j}, \nu_{C_j}), \\
I'_j(\mu_{I'_j}, \nu_{I'_j}) &: \quad \text{If } u \text{ is } A_j(\mu_{A_j}, \nu_{A_j}), \text{ then } w \text{ is } C_j(\mu_{C_j}, \nu_{C_j}), \\
I''_j(\mu_{I''_j}, \nu_{I''_j}) &: \quad \text{If } v \text{ is } B_j(\mu_{B_j}, \nu_{B_j}), \text{ then } w \text{ is } C_j(\mu_{C_j}, \nu_{C_j}).
\end{aligned}$$

The implication I_{j} is the combination of I_{j}' and I_{j}''. If any of the following IFIs are applied: Mamdani IFI: $I_{\mathsf{M}}\ (\mu_{\mathsf{I}_{\mathsf{m}}}, \nu_{\mathsf{I}_{\mathsf{m}}})$, Larsen IFI: $I_{\mathsf{L}}\ (\mu_{\mathsf{I}_{\mathsf{lp}}}, \nu_{\mathsf{I}_{\mathsf{ls}}})$, Bounded IFI: $I_{\mathsf{B}}\ (\mu_{\mathsf{I}_{\mathsf{bp}}}, \nu_{\mathsf{I}_{\mathsf{bs}}})$, Drastic IFI: $I_{\mathsf{D}}\ (\mu_{\mathsf{I}_{\mathsf{dp}}}, \nu_{\mathsf{I}_{\mathsf{ds}}})$, then the following holds:

(1) $(\mu_{\mathsf{A}'}, \mu_{\mathsf{B}'}) \circ \mu_{\mathsf{I}_{\mathsf{j}}} = [\mu_{\mathsf{A}'} \circ \mu_{\mathsf{I}_{\mathsf{j}}'}] \cap [\mu_{\mathsf{B}'} \circ \mu_{\mathsf{I}_{\mathsf{j}}''}]$, if $\mu_{\mathsf{A}_{\mathsf{j}} \times \mathsf{B}_{\mathsf{j}}} = \mu_{\mathsf{A}_{\mathsf{j}}} \wedge \mu_{\mathsf{B}_{\mathsf{j}}} = \min\{\mu_{\mathsf{A}_{\mathsf{j}}}, \mu_{\mathsf{B}_{\mathsf{j}}}\}$, and

(2) $(\nu_{\mathsf{A}'}, \nu_{\mathsf{B}'}) \circ \nu_{\mathsf{I}_{\mathsf{j}}} = [\nu_{\mathsf{A}'} \circ \nu_{\mathsf{I}_{\mathsf{j}}'}] \cap [\nu_{\mathsf{B}'} \circ \nu_{\mathsf{I}_{\mathsf{j}}''}]$, if $\nu_{\mathsf{A}_{\mathsf{j}} \times \mathsf{B}_{\mathsf{j}}} = \nu_{\mathsf{A}_{\mathsf{j}}} \vee \nu_{\mathsf{B}_{\mathsf{j}}} = \max\{\nu_{\mathsf{A}_{\mathsf{j}}}, \nu_{\mathsf{B}_{\mathsf{j}}}\}$.

Proof: The first part (1) follows directly from the result in [31, Lemma 2]. Now, we proceed to the proof of the second part (2), focusing on the non-membership function.

Consider the expression for the non-membership function $\nu_{\mathsf{C}_{\mathsf{j}}'}$:

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = (\nu_{\mathsf{A}'}, \nu_{\mathsf{B}'}) \circ \nu_{\mathsf{I}_{\mathsf{j}}} = (\nu_{\mathsf{A}'}, \nu_{\mathsf{B}'}) \circ (\nu_{\mathsf{A}_{\mathsf{j}} \times \mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}).$$

Substituting the definition of $\nu_{\mathsf{A}_{\mathsf{j}} \times \mathsf{B}_{\mathsf{j}}}$, we get

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = (\nu_{\mathsf{A}'}, \nu_{\mathsf{B}'}) \circ (\max\{\nu_{\mathsf{A}_{\mathsf{j}}}, \nu_{\mathsf{B}_{\mathsf{j}}}\} \to \nu_{\mathsf{C}_{\mathsf{j}}}).$$

For the IFIs $\nu_{\mathsf{I}_{\mathsf{m}}}, \nu_{\mathsf{I}_{\mathsf{ls}}}, \nu_{\mathsf{I}_{\mathsf{bs}}},$ and $\nu_{\mathsf{I}_{\mathsf{ds}}}$, it holds that

$$\max\{\nu_{\mathsf{A}_{\mathsf{j}}}, \nu_{\mathsf{B}_{\mathsf{j}}}\} \to \nu_{\mathsf{C}_{\mathsf{j}}} \text{ is equivalent to } \max\{\nu_{\mathsf{A}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}, \nu_{\mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}\}.$$

Therefore, we can express $\nu_{\mathsf{C}_{\mathsf{j}}'}$ as:

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = \inf_{\mathsf{u;v}} \max\left\{\nu_{\mathsf{A}'}(u), \nu_{\mathsf{B}'}(v), \max\left(\nu_{\mathsf{A}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}, \nu_{\mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}\right)\right\}.$$

This further simplifies to:

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = \max\left\{\inf_{\mathsf{u}} \max\left(\nu_{\mathsf{A}'}(u), \nu_{\mathsf{A}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}\right), \inf_{\mathsf{v}} \max\left(\nu_{\mathsf{B}'}(v), \nu_{\mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}\right)\right\}.$$

Thus, we obtain:

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = \max\left\{\nu_{\mathsf{A}'} \circ (\nu_{\mathsf{A}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}}), \nu_{\mathsf{B}'} \circ (\nu_{\mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}})\right\} = [\nu_{\mathsf{A}'} \circ \nu_{\mathsf{I}_{\mathsf{j}}'}] \cap [\nu_{\mathsf{B}'} \circ \nu_{\mathsf{I}_{\mathsf{j}}''}].$$

□

If the two inputs

$$A' = \left(\mu_{\mathsf{A}'_{\mathsf{x}}}(z) = \begin{cases}\alpha & \text{if } z = x\\ 0 & \text{otherwise}\end{cases}, \quad \nu_{\mathsf{A}'_{\mathsf{x}}}(z) = \begin{cases}\beta & \text{if } z = x\\ 1 & \text{otherwise}\end{cases}\right)$$

and

$$B' = \left(\mu_{\mathsf{B}'_{\mathsf{y}}}(z') = \begin{cases}\alpha' & \text{if } z' = y\\ 0 & \text{otherwise}\end{cases}, \quad \nu_{\mathsf{B}'_{\mathsf{y}}}(z') = \begin{cases}\beta' & \text{if } z' = y\\ 1 & \text{otherwise}\end{cases}\right)$$

are intuitionistic fuzzy singletons defined on $x \in U$ and $y \in V$ respectively, then the output $C_{\mathrm{j}} = (\mu_{\mathsf{C}_{\mathsf{j}}}, \nu_{\mathsf{C}_{\mathsf{j}}})$ is:

(1) $(\alpha^{\wedge} \wedge \mu_{\mathrm{j}}^{\wedge} \wedge \mu_{\mathsf{C}_{\mathsf{j}}}(w),\ \beta^{\vee} \vee \nu_{\mathrm{j}}^{\vee} \vee \nu_{\mathsf{C}_{\mathsf{j}}}(w))$ using Mamdani's intuitionistic fuzzy inference (1),

(2) $(\alpha^{\wedge} \wedge [\mu_{\mathrm{j}}^{\wedge} \cdot \mu_{\mathsf{C}_{\mathsf{j}}}(w)],\ \beta^{\vee} \vee [\nu_{\mathrm{j}}^{\vee} + \nu_{\mathsf{C}_{\mathsf{j}}}(w) - \nu_{\mathrm{j}}^{\vee} \cdot \nu_{\mathsf{C}_{\mathsf{j}}}(w)])$ using Larsen's intuitionistic fuzzy inference (2).

Here, $\alpha^{\wedge} = \min\{\alpha, \alpha'\}$, $\beta^{\vee} = \max\{\beta, \beta'\}$, $\mu_{\mathrm{j}}^{\wedge} = \min\{\mu_{\mathsf{A}_{\mathsf{j}}}(x), \mu_{\mathsf{B}_{\mathsf{j}}}(y)\}$, and $\nu_{\mathrm{j}}^{\vee} = \max\{\nu_{\mathsf{A}_{\mathsf{j}}}(x), \nu_{\mathsf{B}_{\mathsf{j}}}(y)\}$.

Proof: By Theorem , we know that:

$$\nu_{\mathsf{C}_{\mathsf{j}}'} = [\nu_{\mathsf{A}'_{\mathsf{x}}} \circ (\nu_{\mathsf{A}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}})] \cap [\nu_{\mathsf{B}'_{\mathsf{y}}} \circ (\nu_{\mathsf{B}_{\mathsf{j}}} \to \nu_{\mathsf{C}_{\mathsf{j}}})],$$

which implies:

$$\nu_{\mathsf{C}_{\mathsf{j}}'}(w) = \max\left\{\inf_{\mathsf{z}} \max(\nu_{\mathsf{A}'_{\mathsf{x}}}(z), \nu_{\mathsf{A}_{\mathsf{j}}}(z) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)), \inf_{\mathsf{z}'} \max(\nu_{\mathsf{B}'_{\mathsf{y}}}(z'), \nu_{\mathsf{B}_{\mathsf{j}}}(z') \to \nu_{\mathsf{C}_{\mathsf{j}}}(w))\right\}.$$

First, evaluating for A':

$$\inf_{\mathsf{z}} \max(\nu_{\mathsf{A}'_{\mathsf{x}}}(z), \nu_{\mathsf{A}_{\mathsf{j}}}(z) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)) = \max(\beta, \nu_{\mathsf{A}_{\mathsf{j}}}(x) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)).$$

Similarly, for B':

$$\inf_{\mathsf{z}'} \max(\nu_{\mathsf{B}'_{\mathsf{y}}}(z'), \nu_{\mathsf{B}_{\mathsf{j}}}(z') \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)) = \max(\beta', \nu_{\mathsf{B}_{\mathsf{j}}}(y) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)).$$

Thus, combining the results:

$$\nu_{\mathsf{C}_{\mathsf{j}}'}(w) = \max\{\beta, \nu_{\mathsf{A}_{\mathsf{j}}}(x) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w), \beta', \nu_{\mathsf{B}_{\mathsf{j}}}(y) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)\} = \beta^{\vee} \vee \max\{\nu_{\mathsf{A}_{\mathsf{j}}}(x) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w), \nu_{\mathsf{B}_{\mathsf{j}}}(y) \to \nu_{\mathsf{C}_{\mathsf{j}}}(w)\}.$$

Similarly, for the membership function:

$$\mu_{C_j'}(w) = \alpha^{\wedge} \wedge \min\{\mu_{A_j}(x) \to \mu_{C_j}(w), \mu_{B_j}(y) \to \mu_{C_j}(w)\}.$$

(1) Mamdani Intuitionistic Fuzzy Inference: Using Mamdani IFI, the membership and non-membership functions are:

$$\mu_{C_j'}(w) = \alpha^{\wedge} \wedge \min\{\mu_{A_j}(x), \mu_{B_j}(y), \mu_{C_j}(w)\} = \alpha^{\wedge} \wedge \mu_j^{\wedge} \wedge \mu_{C_j}(w),$$
$$\nu_{C_j'}(w) = \beta^{\vee} \vee \max\{\nu_{A_j}(x), \nu_{B_j}(y), \nu_{C_j}(w)\} = \beta^{\vee} \vee \nu_j^{\vee} \vee \nu_{C_j}(w).$$

Therefore, the output is:

$$(\mu_{C'}, \nu_{C'}) = (\alpha^{\wedge} \wedge \mu_j^{\wedge} \wedge \mu_{C_j}(w),\ \beta^{\vee} \vee \nu_j^{\vee} \vee \nu_{C_j}(w)).$$

(2) Larsen Intuitionistic Fuzzy Inference: Using Larsen IFI, the membership and non-membership functions are:

$$\begin{aligned}\mu_{C_j'}(w) &= \alpha^{\wedge} \wedge \min\{\mu_{A_j}(x) \cdot \mu_{C_j}(w), \mu_{B_j}(y) \cdot \mu_{C_j}(w)\} \\ &= \alpha^{\wedge} \wedge [\mu_j^{\wedge} \cdot \mu_{C_j}(w)],\end{aligned}$$

For the non-membership function:

$$\begin{aligned}\nu_{C_j'}(w) &= \beta^{\vee} \vee \max\{\nu_{A_j}(x) + \nu_{C_j}(w) - \nu_{A_j}(x) \cdot \nu_{C_j}(w), \\ &\quad \nu_{B_j}(y) + \nu_{C_j}(w) - \nu_{B_j}(y) \cdot \nu_{C_j}(w)\} \\ &= \beta^{\vee} \vee \left[\nu_j^{\vee} + \nu_{C_j}(w) - \nu_j^{\vee} \cdot \nu_{C_j}(w)\right].\end{aligned}$$

Therefore, the output is:

$$(\mu_{C'}, \nu_{C'}) = (\alpha^{\wedge} \wedge [\mu_j^{\wedge} \cdot \mu_{C_j}(w)], \beta^{\vee} \vee [\nu_j^{\vee} + \nu_{C_j}(w) - \nu_j^{\vee} \cdot \nu_{C_j}(w)]).$$

□

3.1|SISO Intuitionistic Fuzzy Inference Mechanism

In this section, we develop the framework for a Single Input and Single Output (SISO) system within the intuitionistic fuzzy inference mechanism. The following rule establishes the relationship between the input $A^{'}$ and the output $C_j^{'}$, represented by the inutionitic fuzzy set $(\mu_{C_j'}, \nu_{C_j'})$:

if x is A_j then w is C_j

This can be mathematically expressed as: $(A_j) \to C_j$ (Mamdani implication)

3.1.1|Investigation of the Membership Function $\mu_{C_j'}$ for output $C_j^{'}$

The equation (6) is reduced to single input as follows:

$$\mu_{C_j'}(w) = \sup_u \min\{\min\{\mu_{A'}(u), \mu_{A_j}(u)\}, \mu_{C_j}(w)\}.$$

Let $\mu_s := \sup_u \min\{\mu_{A'}(u), \mu_{A_j}(u)\}$ be a membership value of firing strength and it is nothing but $\sup_u\{\mu_{A_j}(u) \mid \mu_{A_j}(u) = \mu_{A'}(u)\}$. Hence,

$$\begin{aligned}\mu_{C_j'}(w) &= \min\{\sup_u \min\{\mu_{A'}(u), \mu_{A_j}(u)\}, \sup_u \mu_{C_j}(w)\} \\ &= \min\{\mu_s, \mu_{C_j}(w)\} = \mu_s \wedge \mu_{C_j}(w) \text{ by definition }.\end{aligned} \tag{7}$$

Let us consider the intuitionistic fuzzy numbers $A = (\mu_{A_j}, \nu_{A_j})$ and input $A^{'} = (\mu_{A'}, \nu_{A'})$ as in the definition . So, we have

$$\mu_{A_j}(x) = \begin{cases} l_{A_j}(x) & \text{if } w_1 \le x \le a \\ 1 & \text{if } a \le x \le b \\ r_{A_j}(x) & \text{if } b \le x \le w_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{A'}(x) = \begin{cases} l_{A'}(x) & \text{if } w_1^{'} \le x \le a^{'} \\ 1 & \text{if } a^{'} \le x \le b^{'} \\ r_{A'}(x) & \text{if } b^{'} \le x \le w_2^{'} \\ 0, & \text{otherwise} \end{cases}$$

are membership functions of intuitionistic fuzzy numbers A and A' respectively. Finding the membership value of the firing strength μ_s is the job at hand in order to obtain the membership value of the output $(\mu_{C_j'})$ for the input $(\mu_{A'})$. In order to find μ_s, one of the following scenarios must be assumed,

Case 1. $[a, b] \cap [a^{'}, b^{'}] \neq \emptyset$. Then $\exists\, x \in [a, b] \cap [a^{'}, b^{'}]$ such that $\mu_{\mathsf{A_j}}(x) = \mu_{\mathsf{A}^{'}}(x) = 1$. Hence,

$$\begin{aligned}\mu_{\mathsf{s}} &= \sup_{\mathsf{u}}\{\min\{\mu_{\mathsf{A}^{'}}(x), mu_{\mathsf{A_j}}(x)\}, \min_{\mathsf{u}\neq\mathsf{x}}\{\mu_{\mathsf{A}^{'}}(u), mu_{\mathsf{A_j}}(u)\}\} \\ &= 1.\end{aligned}$$

Case 2. $[a, b] \cap [a^{'}, b^{'}] = \emptyset$. Now we have two alternative subcases for determining the membership value of firing strength μ_{s}, (i) $a^{'} \leq b^{'} < a \leq b$ and (ii) $a \leq b < a^{'} \leq b^{'}$.

Sub-case 2.1. Let us consider the case, $a^{'} \leq b^{'} < a \leq b$. Suppose $w_2' \leq w_1$, then we can demonstrate

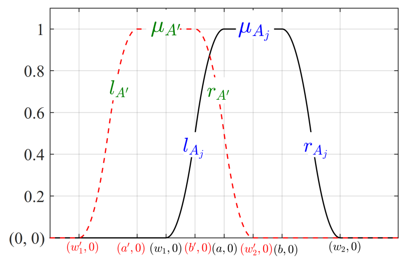

FIGURE 1. Membership Function of IFN under the Condition $a^{'} \leq b^{'} < a \leq b$.

$\mu_{\mathsf{s}} = 0$, by definition of μ_{s}. When $w_2' > w_1$ see Fig. 1, then we consider $H = \{x \in [w_1, w_2'] \mid \mu_{\mathsf{A_j}}(x) = \mu_{\mathsf{A}^{'}}(x)\}$ and H is non-empty because $b^{'} < a$. By our assumptions $a^{'} \leq b^{'} < a \leq b$ and $w_2' > w_1$, there exist sets H_1 and H_2 such that $H_1 = \{x \in [w_1, a) \mid l_{\mathsf{A_j}}(x) = r_{\mathsf{A}^{'}}(x)\} \neq \emptyset$, $H_2 = \{x \in (b\ w'_2] \mid r_{\mathsf{A_j}}(x) = r_{\mathsf{A}^{'}}(x)\}$ and $H = H_1 \cup H_2$. Here H is closed and bounded in $\mathbb{R}$. Hence H is compact by Heine-Borel theorem. Since $\mu_{\mathsf{A_j}}$ is continuous, $\mu_{\mathsf{A_j}}$ has maximum value on H itself. This implies there exist $w \in H$ such that

$$\begin{aligned}\mu_{\mathsf{A_j}}(w) &= \sup\{\mu_{\mathsf{A_j}}(u) | u \in H_1 \cup H_2\} \\ &= \max\{\sup\{\mu_{\mathsf{A_j}}(u) | u \in H_1\}, \sup\{\mu_{\mathsf{A_j}}(u) | u \in H_2\}\}.\end{aligned}$$

Since $r_{\mathsf{A}^{'}}$ is monotonically decreasing and $l_{\mathsf{A_j}}$ is monotonically increasing, there exist constant $p \in [0, 1]$ such that $r_{\mathsf{A}^{'}}(x) = l_{\mathsf{A_j}}(x) = p$ for all $x \in H_1$. Since $r_{\mathsf{A}^{'}}$ and $r_{\mathsf{A_j}}$ are monotonically decreasing, $p \geq \sup\{\mu_{\mathsf{A_j}}(u) | u \in H_2\}$. Clearly $\mu_{\mathsf{A_j}}(w) = \mu_{\mathsf{s}}$ and thereby

$$\mu_{\mathsf{s}} = \max\{p, \sup\{\mu_{\mathsf{A_j}}(u) | u \in H_2\}\} = p.$$

Subcase 2.2. Let us consider the case, $a \leq b < a' \leq b'$. Similarly as subcase 2.1, we can obtain the membership value

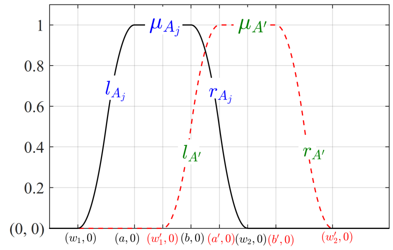

FIGURE 2. Membership Function of IFN under the Condition $a \leq b < a' \leq b'$.

of firing strength μ_{s} but here $H = \{x \in [w'_1, w_2] \mid \mu_{\mathsf{A_j}}(x) = \mu_{\mathsf{A}^{'}}(x)\} = H_1 \cup H_2$ where $H_1 = \{x \in (b\ w_2] \mid r_{\mathsf{A_j}}(x) = l_{\mathsf{A}^{'}}(x)\} \neq \emptyset$ and $H_2 = \{x \in [w_1^{'}, a) \mid l_{\mathsf{A_j}}(x) = l_{\mathsf{A}^{'}}(x)\}$. Now with the aforementioned information, we will explore membership function $\mu_{\mathsf{C_j}^{'}}$ of output $C_{\mathsf{j}}^{'}$ as follows. Let us consider the membership function
@XX@XX@

$$\mu_{\mathsf{C_j}}(x) = \begin{cases} l_{\mathsf{C_j}}(x) & \text{if } g_1 \leq x \leq p \\ 1 & \text{if } p \leq x \leq q \\ r_{\mathsf{C_j}}(x) & \text{if } q \leq x \leq g_2 \\ 0, & \text{otherwise} \end{cases}.$$

of the intuitionistic fuzzy number C_{j} as definition . Let us consider the set $I = \{x \in \mathbb{R} \mid \mu_{\mathsf{C_j}}(x) \geq \mu_{\mathsf{s}}\}$ which is nothing but $\mu_{\mathsf{C_j}}^{-1}[\mu_{\mathsf{s}}\ 1]$. Here I will be closed interval in $\mathbb{R}$ because $\mu_{\mathsf{C_j}}$ is membership function of intuitionistic fuzzy number C_{j}

and hence μ_{C_j} is continuous function. Hence there exist $x^*, y^* \in \mathbb{R}$ such that $I = [x^*, y^*]$. According to equation (7), the output for input $\mu_{A'}$ is

$$\mu_{C'_j}(w) = \mu_s \wedge \mu_{C_j}(w).$$

If $w \in [x^*, y^*]$, then $\mu_{C_j}(w) \geq \mu_s$. This implies $\mu_{C'_j}(w) = \mu_s$. If $w \in [g_1, x^*) \cup (y^*, g_2]$, then $\mu_{C_j}(w) < \mu_s$. Therefore $\mu_{C'_j}(w) = \mu_{C_j}(w)$ and the output is

$$\mu_{C'_j}(w) = \begin{cases} l_{C_j}(w) & \text{if } g_1 \leq w \leq x^* \\ \mu_s & \text{if } x^* \leq w \leq y^* \\ r_{C_j}(w) & \text{if } y^* \leq w \leq g_2 \\ 0, & \text{otherwise} \end{cases}. \tag{8}$$

3.1.2|Investigation of the Non-membership Function $\nu_{C'_j}$ for output C'_j

The equation (6) is reduced to single input as follows:

$$\nu_{C'_j}(w) = \inf_v \max\{\max\{\nu_{A'}(v), nu_{A_j}(v)\}, \nu_{C_j}(w)\}.$$

Let us consider $\nu_s := \inf_v \max\{\nu_{A'}(v), nu_{A_j}(v)\}$ is non-membership value of firing strength and it is nothing but $\inf_v\{\nu_{A_j}(v) \mid \nu_{A_j}(v) = \nu_{A'}(v)\}$. Hence,

$$\begin{aligned} \nu_{C'_j}(w) &= \max\{\inf_v \max\{\nu_{A'}(v), nu_{A_j}(v)\}, \inf_v \nu_{C_j}(w)\} \\ &= \max\{\nu_s, \nu_{C_j}(w)\} = \nu_s \vee \nu_{C_j}(w) \text{ by definition ..} \end{aligned} \tag{9}$$

Let

$$\nu_{A_j}(y) = \begin{cases} l'_{A_j}(y) & \text{if } u_1 \leq y \leq c \\ 0 & \text{if } c \leq y \leq d \\ r'_{A_j}(y) & \text{if } d \leq y \leq u_2 \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{A'}(y) = \begin{cases} l'_{A'}(y) & \text{if } u'_1 \leq y \leq c' \\ 0 & \text{if } c' \leq y \leq d' \\ r'_{A'}(y) & \text{if } d' \leq y \leq u'_2 \\ 1, & \text{otherwise} \end{cases}$$

be non-membership functions of intuitionistic fuzzy numbers A and A' as definition , To obtain the non-membership value of output $\nu_{C'_j}$ for input $\nu_{A'}$ our task is to determine the non-membership value of firing strength ν_s. To find ν_s, we must assume one of the following scenarios,

Case 1. $[c, d] \cap [c', d'] \neq \emptyset$. Then $\exists\, y \in [c, d] \cap [a', b']$ such that $\nu_{A_j}(y) = \nu_{A'}(y) = 0$. Hence,

$$\begin{aligned} \nu_s &= \inf_v\{\max\{\nu_{A'}(y), \nu_{A_j}(y)\}, \max_{v \neq y}\{\nu_{A'}(v), \nu_{A_j}(v)\}\} \\ &= 0. \end{aligned}$$

Case 2. $[c, d] \cap [c', d'] = \emptyset$. Here there are two possible subcases for obtaining non-membership value of firing strength ν_s, which are (i) $c' \leq d' < c \leq d$ and (ii) $c \leq d < c' \leq d'$.

Subcase 2.1. Let us consider the case, $c' \leq d' < c \leq d$. If $u'_2 \leq u_1$, then we can show $\nu_s = 1$ by defini-

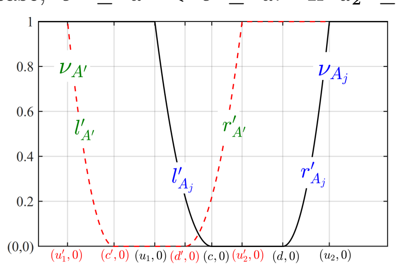

FIGURE 3. Non - Membership Function of IFN under the Condition $c' \quad d' < d$.

tion of νs. When $u'_2 > u_1$ see Fig. 3, then let us consider $K = \{y \in [u_1, u'_2] \mid \nu Aj(y) = \nu' A\ (y)\}$ and K is non-empty because $d' < c$. We can find sets K_1 and K_2 such that $K_1 = \{y \in [u_1, c) \mid l' \ (y) = r' \ ' (y)\} \neq \emptyset$,

$K_2 = \{y \in (d\ u'_2] \mid r'_{\mathsf{A_j}}(y) = r'_{\mathsf{A}'}(y)\}$ and $K = K_1 \cup K_2$. K is compact by closed and bounded in $\mathbb{R}$. Since $\nu_{\mathsf{A_j}}$ is continuous, $\nu_{\mathsf{A_j}}$ has minimum value on K itself. This implies there exist $w \in K$ such that

$$\begin{aligned}\nu_{\mathsf{A_j}}(w) &= \inf\{\nu_{\mathsf{A_j}}(v)|v \in K\} = \inf\{\nu_{\mathsf{A_j}}(v)|v \in K_1 \cup K_2\} \\ &= \min\{\inf\{\nu_{\mathsf{A_j}}(v)|v \in K_1\}, \inf\{\nu_{\mathsf{A_j}}(v)|v \in K_2\}\}.\end{aligned}$$

Since $r'_{\mathsf{A}'}$ is monotonically increasing and $l'_{\mathsf{A_j}}$ is monotonically decreasing, there exist constant $q \in [0,1]$ such that $r'_{\mathsf{A}'}(y) = l'_{\mathsf{A_j}}(y) = q$ for all $x \in H_1$. Due to $r'_{\mathsf{A}'}$ and $r'_{\mathsf{A_j}}$ are monotonically increasing we obtain $q \leq \inf\{\nu_{\mathsf{A_j}}(v)|u \in H_2\}$. Clearly $\nu_{\mathsf{A_j}}(w) = \nu_{\mathsf{s}}$ and consequently

$$\nu_{\mathsf{s}} = \min\{q, \inf\{\nu_{\mathsf{A_j}}(v)|u \in H_2\}\} = q.$$

Subcase 2.2. Let us consider the case, $c \leq d < c' \leq d'$. We can derive the non-membership value of firing strength

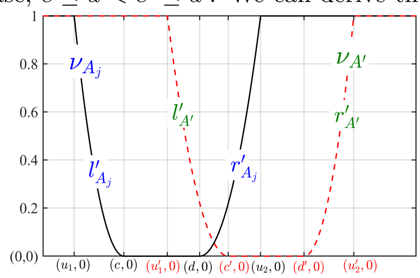

FIGURE 4. Non - Membership Function of IFN under the Condition $c \leq d < c' \leq d'$.

ν_{s} in the same way as in subcase $(2(i))$, but here $K = \{y \in [u'_1, u_2] \mid \nu_{\mathsf{A_j}}(y) = \nu_{\mathsf{A}'}(y)\} = K_1 \cup K_2$ where $K_1 = \{y \in (d\ u_2] \mid r'_{\mathsf{A_j}}(y) = l'_{\mathsf{A}'}(y)\} \neq \emptyset$ and $K_2 = \{x \in [u'_1, c) \mid l'_{\mathsf{A_j}}(y) = l'_{\mathsf{A}'}(y)\}$. Following the information above, we will analyses non-membership function $\nu_{\mathsf{C}'_{\mathsf{j}}}$ of output C'_{j} as follows. Let us consider

$$\nu_{\mathsf{C_j}}(y) = \begin{cases} l'_{\mathsf{C_j}}(y) & \text{if } h_1 \leq y \leq r \\ 0 & \text{if } r \leq y \leq s \\ r'_{\mathsf{C_j}}(y) & \text{if } s \leq y \leq h_2 \\ 1, & \text{otherwise} \end{cases}.$$

be a non-membership function of intuitionistic fuzzy number C_{j} as defined in . Let us consider the set $L = \{y \in \mathbb{R} \mid \nu_{\mathsf{C_j}}(y) \leq \nu_{\mathsf{s}}\}$ which is nothing but $\nu_{\mathsf{C_j}}^{-1}[0,;\nu_{\mathsf{s}}]$. Here L will be closed interval in $\mathbb{R}$ because $\nu_{\mathsf{C_j}}$ is non-membership function of intuitionistic fuzzy number C_{j} and hence $\nu_{\mathsf{C_j}}$ is continuous function. Hence there exist $m^*, n^* \in \mathbb{R}$ such that $L = [m^*, n^*]$. The output for the input $\nu_{\mathsf{A}'}$ is

$$\nu_{\mathsf{C}'_{\mathsf{j}}}(w) = \nu_{\mathsf{s}} \vee \nu_{\mathsf{C_j}}(w)$$

according to equation (9). If $w \in [m^*, n^*]$, then $\nu_{\mathsf{C_j}}(w) \leq \nu_{\mathsf{s}}$. This implies $\nu_{\mathsf{C}'_{\mathsf{j}}}(w) = \nu_{\mathsf{s}}$. If $w \in [h_1, m^*) \cup (n^*\ h_2]$, then $\nu_{\mathsf{C_j}}(w) > \nu_{\mathsf{s}}$. Therefore $\nu_{\mathsf{C}'_{\mathsf{j}}}(w) = \nu_{\mathsf{C_j}}(w)$ and output is

$$\nu_{\mathsf{C}'_{\mathsf{j}}}(w) = \begin{cases} l'_{\mathsf{C_j}}(w) & \text{if } h_1 \leq w \leq m^* \\ \nu_{\mathsf{s}} & \text{if } m^* \leq w \leq n^* \\ r'_{\mathsf{C_j}}(w) & \text{if } n^* \leq w \leq h_2 \\ 1, & \text{otherwise} \end{cases}.$$

Hence the required Output C'_{j} for input A' is

$$C'_{\mathsf{j}}(w) = (\mu_{\mathsf{C}'_{\mathsf{j}}}(w), nu_{\mathsf{C}'_{\mathsf{j}}}(w)).$$

Here $(\mu_{\mathsf{s}}, \nu_{\mathsf{s}})$ is known as weight of premise.

3.2| MISO Intuitionistic Fuzzy Inference Mechanism

In this section, we present the Multi-Input Single-Output (MISO) Intuitionistic Fuzzy Inference Mechanism. Consider an intuitionistic fuzzy rule-based system where the inputs and the output follow a multi-input and single-output (MISO) structure. Let the inputs be denoted as (A', B'), and the output is denoted as $C_j'\ (\mu_{C_j'}, \nu_{C_j'})$,. The rule-based system can be represented by the following inference implication:

$$\mathbf{I_j}: \text{ if } x \text{ is } A_j \text{ and } y \text{ is } B_j \text{ then } w \text{ is } C_j$$

where A_j, B_j and C_j are intuitionistic fuzzy numbers defined on the universes U, V and W respectively. The corresponding membership and non-membership functions for the intuitionistic fuzzy rule can be expressed using the Mamdani implication, as follows:

$$(\mu_{A_j \times B_j}, \nu_{A_j \times B_j}):\ (\mu_{A_j} \wedge \mu_{B_j}, \nu_{A_j} \vee \nu_{B_j})$$

Determination of Membership Functions $\mu_{\mathbf{C_j'}}(\mathbf{w})$

Our objective is to determine the membership μ_s and non-membership ν_s values for the output C_j'. According to Theorem , the membership function $\mu_{C_j'}$ for the output can be computed as:

$$\mu_{C_j'} = \mu_{C_{j_A}'} \cap \mu_{C_{j_B}'}$$

where $\mu_{C_{j_A}'} = \mu_{A'} \circ \mu_{I_j'}$ and $\mu_{C_{j_B}'} = \mu_{B'} \circ \mu_{I_j''}$.

In this case, both $\mu_{C_{j_A}'}$ and $\mu_{C_{j_B}'}$ follow a Single Input Single Output (SISO) behavior. This implies that the intermediate membership functions μ_{s_A} and μ_{s_B} can be obtained as:

$$\mu_{C_{j_A}'}(w) = \mu_{s_A} \wedge \mu_{C_j}(w) \quad \text{and} \quad \mu_{C_{j_B}'}(w) = \mu_{s_B} \wedge \mu_{C_j}(w)$$

Thus, the overall membership function for the MISO system becomes:

$$\mu_{C_j'}(w) = \mu_s \wedge \mu_{C_j}(w), \tag{10}$$

where $\mu_s = \mu_{s_A} \wedge \mu_{s_B}$.

Determination of Non-Membership Functions $\nu_{\mathbf{C_j'}}(\mathbf{w})$

Similarly, the non-membership function $\nu_{C_j'}$ is derived using the same principle, following Theorem . The expression for $\nu_{C_j'}$ is given by:

$$\nu_{C_j'} = \nu_{C_{j_A}'} \cap \nu_{C_{j_B}'}$$

where $\nu_{C_{j_A}'} = \nu_{A'} \circ \nu_{I_j'}$ and $\nu_{C_{j_B}'} = \nu_{B'} \circ \nu_{I_j''}$. Here, $\nu_{C_{j_A}'}$ and $\nu_{C_{j_B}'}$ follow a SISO behavior. Thus, the non-membership values ν_{s_A} and ν_{s_B} are determined as:

$$\nu_{C_{j_A}'}(w) = \nu_{s_A} \vee \nu_{C_j}(w) \quad \text{and} \quad \nu_{C_{j_B}'}(w) = \nu_{s_B} \vee \nu_{C_j}(w)$$

Hence, the overall non-membership function for the MISO system is expressed as:

$$\nu_{C_j'}(w) = \nu_s \vee \nu_{C_j}(w), \tag{11}$$

where $\nu_s = \nu_{s_A} \vee \nu_{s_B}$. Consequently, the final expressions for the membership and non-membership functions of the output C_j' in the MISO system are:

$$(\mu_{C_j'}(w), \nu_{C_j'}(w)) = (\mu_s \wedge \mu_{C_j}(w), \nu_s \vee \nu_{C_j}(w))$$

It holds that: $0 \leq \mu_s + \nu_s \leq 1$.

4|Defuzzification in Intuitionistic Fuzzy Control Systems

Defuzzification is a critical phase in intuitionistic fuzzy control systems, facilitating the conversion of intuitionistic fuzzy outputs into crisp numerical values. This process enhances the comprehensibility of results for experts and practitioners, enabling them to make decisions based on the output of the system.

For a given output $C_j^{'}(w) = (\mu_{C_j'}(w), \nu_{C_j'}(w))$, where $\mu_{C_j'}(w)$ and $\nu_{C_j'}(w)$ denote the membership and non-membership functions, respectively, we define the intuitionistic defuzzification operator w as follows:

$$w = \frac{\int_W w \cdot [\Theta \cdot \mu_{C_j'}(w) \ + \ (1-\Theta)\cdot(1-\nu_{C_j'}(w))]\, dw}{\int_W [\Theta \cdot \mu_{C_j'}(w) \ + \ (1-\Theta)\cdot(1-\nu_{C_j'}(w))]\, dw} . \tag{12}$$

In this equation, Θ is a parameter that varies within the interval $[0,1]$. The selected value of Θ plays a significant role in determining the output of the defuzzification process. When Θ is closer to 1, the emphasis is placed more heavily on the membership function $\mu_{C_j'}(w))$, thereby reducing the influence of hesitancy. This means that the resulting crisp output will be more aligned with the degree of membership in the fuzzy set, which is useful in scenarios where certainty is preferred. Conversely, when Θ is closer to 0, the non-membership function $\nu_{C_j'}(w)$ gains prominence, increasing the impact of hesitancy in the defuzzification process. This is valuable in cases where ambiguity or uncertainty is prevalent, allowing for a more cautious interpretation of the output. The relationship between Θ and the significance of hesitancy can be mathematically expressed as

$$\pi_{C_j'} \propto \frac{1}{\Theta},$$

where $\pi_{C_j'} = 1 - \mu_{C_j'} - \nu_{C_j'}$ Thus, an increase in Θ results in a decrease in the hesitancy component, reflecting a preference for more decisive outputs. The flexibility in adjusting Θ allows practitioners to tailor the defuzzification process according to specific application contexts. For example, in high-stakes decision-making scenarios, one might choose a higher Θ to prioritize confidence over ambiguity. In contrast, in exploratory or experimental situations, a lower Θ might be selected to account for potential uncertainties. Notably, when $\Theta = 1$, the defuzzification process simplifies to the Center of Gravity (**COG**) approach commonly used in fuzzy systems. This connection highlights the versatility of the defuzzification operator w , allowing it to bridge traditional fuzzy logic with intuitionistic fuzzy logic frameworks.

4.1 |Defuzzification for Trapezoidal Intuitionistic Fuzzy Numbers

Let $A = [(a,\ b,\ c,\ d), (e,\ f,\ g,\ h)]$ be trapezoidal intuitionistic fuzzy number with the condition

$$a \neq b \neq c \neq d \neq e \neq f \neq g \neq h \tag{13}$$

According to the definition , the foot order is $e \leq a, f \leq b \leq c \leq g, d \leq h$. Thus, the potential orders of the foot for the trapezoidal intuitionistic fuzzy number A can be represented as follows:

$$\begin{aligned} &i)\ e < a < f < b < c < g < d < h,\\ &ii)\ e < f < a < b < c < d < g < h,\\ &iii)\ e < f < a < b < c < g < d < h,\\ &iv)\ e < a < f < b < c < d < g < h \end{aligned}$$

Let us consider the case:

$$e < a < f < b < c < g < d < h. \tag{14}$$

Since the equation of defuzzification (12) is given by

$$w = \frac{\int_W w \cdot [\Theta \cdot \mu_A(w) \ + \ (1-\Theta)\cdot(1-\nu_A(w))]\, dw}{\int_W [\Theta \cdot \mu_A(w) \ + \ (1-\Theta)\cdot(1-\nu_A(w))]\, dw}, \tag{15}$$

w represents the crisp value for $A = [(a,\ b,\ c,\ d), (e,\ f,\ g,\ h)]$ when $\Theta \in [0,1]$. Let $N(w) = w \cdot [\Theta \cdot \mu_A(w) \ + \ (1-\Theta) \cdot (1-\nu_A(w))]$ and $D(w) = \Theta \cdot \mu_A(w) \ + \ (1-\Theta)\cdot(1-\nu_A(w))$. Therefore,

$$\begin{aligned} \int_W N(w)\, dw &= \int_e^a N(w)\, dw + \int_a^f N(w)\, dw + \int_f^b N(w)\, dw + \int_b^c N(w)\, dw + \int_c^g N(w)\, dw + \int_g^d N(w)\, dw + \int_d^h N(w)\, dw \\ &= \frac{1}{6}\Bigg(\Theta\Big(\frac{1}{b-a}[a^3 - 3ab^2 + 2b^3] + \frac{1}{d-c}[d^3 - 3dc^2 + 2c^3]\Big) + (1-\Theta)\Big(\frac{1}{f-e}[e^3 - 3ef^2 + 2f^3] \\ &\quad + \frac{1}{h-g}[h^3 - 3hg^2 + 2g^3] + 3[g^2 - f^2 + b^2 - c^2]\Big) + 3[c^2 - b^2]\Bigg) \end{aligned} \tag{16}$$

and

$$\int_W D(w)\,dw = \int_e^a D(w)\,dw + \int_a^f D(w)\,dw + \int_f^b D(w)\,dw + \int_b^c D(w)\,dw + \int_c^g D(w)\,dw + \int_g^d D(w)\,dw + \int_d^h D(w)\,dw$$
$$= \frac{1}{2}\Bigg(\Theta[b-a+d-c] + [1-\Theta][2(b-c)+h-e+g-f] + 2[c-b]\Bigg). \tag{17}$$

By substituting equations (16) and (17) into equation (15), the defuzzified value w^Θ can be obtained.

In a similar fashion, w^Θ for $[(a,\ b,\ c,\ d),(e,\ f,\ g,\ h)]$ can be derived for each case listed in equation (13): $\{b := a\}, \{d := c\}, \{e := f\}, \{h := g\}, \{b := a, d := c\}, \{b := a, e := f\}, \{b := a, h := g\}, \{d := c, e := f\}, \{d := c, h := g\}, \{e := f, h := g\}, \{b := a, d := c, e := f\}, \{b := a, d := c, h := g\}, \{b := a, e := f, h := g\}, \{d := c, e := f, h := g\}$ and $\{b := a, d := c, e := f, h := g\}$. Now, if we substitute $\{b := a\}$ in (13), we can derive the trapezoidal intuitionistic fuzzy number $[(a,\ b,\ c,\ d),(e,\ f,\ g,\ h)]$ under the condition that $a = b \neq c \neq d \neq e \neq f \neq g \neq h$. Hence, equation (14) transforms into

$$e < a = f = b < c < g < d < h.$$

Consequently, we have:

$$w^\Theta = \frac{\Bigg(\frac{\Theta}{d-c}[d^3 - 3dc^2 + 2c^3] + (1-\Theta)\Bigg(\frac{1}{(a-e)}[e^3 - 3ea^2 + 2a^3] + \frac{1}{h-g}[h^3 - 3hg^2 + 2g^3] + 3[g^2 - c^2]\Bigg) + 3[c^2 - a^2]\Bigg)}{3\Bigg(\Theta[d-c] + [1-\Theta][a+h-e+g-2c] + 2[c-a]\Bigg)}. \tag{18}$$

Thus, w^Θ is the crisp value obtained for the trapezoidal intuitionistic fuzzy number $[(a,\ b,\ c,\ d),(e,\ f,\ g,\ h)]$. By employing this method, we can determine w^Θ for any configuration of trapezoidal intuitionistic fuzzy numbers. Let us consider the triangular fuzzy number $(l,\ m,\ n)$. In this case, we have $a = l, b = c = m, d = n$ and $e = l, f = g = m, h = l$. Consequently, Θ will be equal to 1 due to the inherent fuzziness. Thus, the defuzzification output w^Θ for the triangular fuzzy number $(l,\ m,\ n)$ is

$$\frac{1 \cdot \Bigg(\frac{1}{m-l}[l^3 - 3lm^2 + 2m^3] + \frac{1}{n-m}[n^3 - 3nm^2 + 2m^3]\Bigg)}{3(n-l)} = \frac{(l+m+n)}{3},$$

which corresponds precisely to the center of gravity (**COG**) of the triangular fuzzy number $(l,\ m,\ n)$ as expected which shows the proposed method of defuzzification of Trapezoidal intuitionistic fuzzy numbers is extending the existing standard defuzzification method by the center of gravity (**COG**) in fuzzy set up into intuitionistic fuzzy set up.

4.2 |Significance of the Proposed Method

The significance of our proposed method is illustrated in Table 1, which presents a comparative analysis between our approach and several existing techniques. This comparison highlights the shortcomings of the existing methods and underscores the advantages of our proposed method. Table 1 provides a comprehensive evaluation of the proposed method against these established techniques. The examples in the table are considered as Intuitionistic Fuzzy Multi-Criteria Decision-Making (IFMCDM) problems, featuring two alternatives evaluated under a single criterion, thereby facilitating a thorough comparison.

For instance, let us consider two different Trapezoidal Intuitionistic Fuzzy Numbers : $A = [(0.2, 0.3, 0.5, 0.6), (0.0, 0.1, 0.6, 0.7)]$ and $A' = [(0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6)]$. Applying the method proposed by **Nehi** [39], we derive the following results: $C^k_\mu(A) = C^k_\mu(A') = 0.4$ and $C^k_\nu(A) = C^k_\nu(A') = 0.360 \Rightarrow A = A'$. This outcome is logically inconsistent, as A and A' are distinctly different trapezoidal intuitionistic fuzzy numbers. In contrast, when we apply our proposed method to A and A', we obtain: $w^\Theta(A) = 0.367 > 0.360 = w^\Theta(A') \Rightarrow A > A'$. This result demonstrates that our proposed method is superior when compared to Nehi's method. The significance of our method is similarly validated through comparisons with other existing techniques, as shown in Table 1. Each entry illustrates the limitations faced by traditional approaches while reinforcing the effectiveness of our proposed method.

TABLE 1. Significance of Proposed Defuzzification Method

Existing Methods	**Limitations of Existing Methods**	**Numerical Example**	**Proposed Method**
Nehi, H.M., [39] $C^{k}(A) = \frac{b+c}{2} + \frac{a-b+d-c}{2(k+2)}, k \in \mathbb{R}^+$ $C^{k}(A) = \frac{e+h}{2} + \frac{f-e+g-h}{2(k+2)}, k \in \mathbb{R}^+$	$A = [(a,b,c,d),(e,f,g,h)]$ $A' = [(a+\epsilon, b+\epsilon, c-\epsilon, d-\epsilon),$ $(e+\epsilon, f+\epsilon, g-\epsilon, h-\epsilon)]$ $C^{k}(A) = C^{k}(A')$ and $C^{k}(A) = C^{k}(A') \Rightarrow A = A'$	$A = [(0.2,0.3,0.5,0.6),(0.0,0.1,0.6,0.7)]$ $A' = [(0.3,0.4,0.4,0.5),(0.1,0.2,0.5,0.6)]$ $C^{k}(A) = C^{k}(A') = 0.4$ $C^{k}(A) = C^{k}(A') = 0.360$ $\Rightarrow A = A'$	$w\ \ (A) = 0.367$ $w\ \ (A') = 0.360$ $A > A'$
Jun Ye [53] $EV(A) = \frac{a+b+c+d+e+f+g+h}{8}$	$A = [(a,b,c,d),(e,f,g,h)]$ $A' = [(a+\epsilon, b+\epsilon, c+\epsilon, d+\epsilon),$ $(e-\epsilon, f-\epsilon, g-\epsilon, h-\epsilon)]$ $EV(A) = EV(A') \Rightarrow A = A'$	$A = [(0.1,0.2,0.3,0.4),(0.1,0.15,0.55,0.6)]$ $A' = [(0.2,0.3,0.4,0.5),(0,0.05,0.45,0.5)]$ $EV(A) = EV(A') = 0.3$ $\Rightarrow A = A'$	$w\ \ (A) = 0.319$ $w\ \ (A') = 0.281$ $A > A'$
De Debaroti Das, P.K., [17] $V(A_1) = V\ \ (A_1) + \lambda(V\ \ (A_1) - V\ \ (A_1))$ $A(A_1) = A\ \ (A_1) + \lambda(A\ \ (A_1) - A\ \ (A_1))$	$A_1 = [(a,b,c,d),(a,b,c,d)]$ $A_1' = [(a-\epsilon, b+\frac{\epsilon}{2}, c+\frac{\epsilon}{2}, d-\epsilon),$ $(a-\epsilon, b+\frac{\epsilon}{2}, c+\frac{\epsilon}{2}, d-\epsilon)]$ $V(A_1) = V(A_1')$ and $A(A_1) = V(A_1') \Rightarrow A_1 = A_1'$	$A_1 = [(0.2,0.35,0.4,0.55),(0.2,0.35,0.4,0.55)]$ $A_1' = [(0.1,0.4,0.45,0.45),(0.1,0.4,0.45,0.45)]$ $V(A_1) = V(A_1') = 0.233$ $A(A_1) = V(A_1') = 0.083$ $\Rightarrow A_1 = A_1'$	$w\ \ (A_1) = 0.375$ $w\ \ (A_1') = 0.331$ $A_1 > A_1'$
Kumar, A., Kaur, M., [27] $M^{;k}\ \ (A) = \frac{\beta(b+c)}{2} + \frac{(1-\beta)(a+d)}{2} + \frac{(1-\beta)(b-a+c-d)(k+1)}{2(k+1)}$ $M^{;k}\ \ (A) = \frac{\beta(b'+c')}{2} + \frac{(1-\beta)(a'+d')}{2} + \frac{(1-\beta)(b'-a'+c'-d')(k+1)}{2(k+1)},$ $k \in \mathbb{R}^+$ and $\beta \in [0,1]$	$A = [(a,b,c,d),(e,f,g,h)]$ $A' = [(a-\epsilon, b-\epsilon, c+\epsilon, d+\epsilon),$ $(e-\epsilon, f-\epsilon, g+\epsilon, h+\epsilon)]$ $M^{;k}\ \ (A) = M^{;k}\ \ (A')$ and $M^{;k}\ \ (A) = M^{;k}\ \ (A') \Rightarrow A = A'$	$A = [(0.2,0.3,0.5,0.6),(0.0,0.1,0.6,0.7)]$ $A' = [(0.2,0.3,0.4,0.5),(0,0.05,0.45,0.5)]$ $M^{;k}\ \ (A) = M^{;k}\ \ (A') = 0.4$ $M^{;k}\ \ (A) = M^{;k}\ \ (A') = 0.35$ $\Rightarrow A = A'$	$w\ \ (A) = 0.367$ $w\ \ (A') = 0.360$ $A > A'$

5|Transitioning from Trapezoidal Fuzzy Numbers to Trapezoidal Intuitionistic Fuzzy Numbers with α - Hesitancy (Intuitionistic Fuzzification)

This section discusses the transformation of trapezoidal fuzzy numbers characterized by α - hesitancy into trapezoidal intuitionistic fuzzy numbers. This transformation is particularly relevant in scenarios where uncertainty and ambiguity are inherent in decision-making processes. To illustrate this concept, consider a practical example involving an individual u who owns a vegetable shop in a village. Let $M(u)$ and $N(u)$ denote the percentages of residents in the village who support and oppose the operation of shop u, respectively. Additionally, there may be residents who are ambivalent, neither supporting nor opposing the shop; this percentage is represented as $H(u)$. In this context, the operation of shops in the village can be framed within an intuitionistic fuzzy set framework. The membership function $\mu_{A}(u)$, non-membership function $\nu_{A}(u)$, and hesitancy degree $\pi_{A}(u)$ are defined as follows:

$$\mu_{A}(u) = \frac{M(u)}{100}, \nu_{A}(u) = \frac{N(u)}{100}, \pi_{A}(u) = \frac{H(u)}{100}.$$

Here, $\pi_{A}(u)$ quantifies the degree of hesitancy or uncertainty regarding the acceptance of u within the context of running the shop. The decision to utilize intuitionistic fuzzy numbers rather than traditional fuzzy numbers acknowledges that

the level of hesitancy can vary among experts. It is crucial to recognize that an expert will not typically collect data with 100% hesitancy; thus, if an expert gathers data with h% hesitancy, we can express this uncertainty as $\alpha = \frac{h}{100}$.

5.1 |Theoretical Justification

The theoretical underpinnings for this transformation are grounded in intuitionistic fuzzy set theory, introduced by Atanassov. This theory extends classical fuzzy sets by incorporating both membership and non-membership degrees, which allows for a more nuanced representation of uncertainty. Traditional fuzzy sets are limited in their ability to capture the complexities of human judgment, especially when opinions are not strictly supportive or oppositional. The introduction of hesitancy as a third component provides a more comprehensive model for situations characterized by ambiguity. The transformation from trapezoidal fuzzy numbers to trapezoidal intuitionistic fuzzy numbers can be mathematically formalized. We define a trapezoidal intuitionistic fuzzy number with α-level hesitancy derived from a trapezoidal fuzzy number as follows:

$$\left[(a,\ b,\ c,\ d),(a^{\alpha+1},\ b^{\alpha+1},\ \frac{c+\alpha}{\alpha+1},\ \frac{d+\alpha}{\alpha+1})\right]$$

for $\alpha \in [0,\ 1]$ where $a, b, c, d \in [0,\ 1]$. The conditions $a^{\alpha+1} \leq a,\ b^{\alpha+1} \leq b \leq c \leq \frac{c+\alpha}{\alpha+1},\ d \leq \frac{d+\alpha}{\alpha+1}$ ensure the preservation of ordering and logical consistency among the parameters. Notably, if $\alpha = 0$, this indicates that the expert experiences no hesitancy, which corresponds to a traditional fuzzy environment result the original trapezoidal fuzzy number $(a,\ b,\ c,\ d)$. Conversely, as α approaches 1, the representation accommodates increased uncertainty, reflecting a more nuanced understanding of expert opinion in situations where data is inherently ambiguous or imprecise.

5.2 |Practical Justification

In practical decision-making processes, particularly those involving complex human opinions and uncertainty, capturing the nuances of support, opposition, and hesitancy is critical for effective outcomes. By transitioning from trapezoidal fuzzy numbers with α-hesitancy to trapezoidal intuitionistic fuzzy numbers, decision-makers can enhance their models to better represent real-world complexities. Traditional fuzzy sets, which only account for membership and non-membership values, may not sufficiently handle situations where individuals or experts exhibit hesitancy or ambiguity. Intuitionistic fuzzy sets provide a more comprehensive framework by incorporating a third dimension—hesitancy—alongside membership and non-membership values. This allows for better representation of undecided or ambivalent opinions in scenarios like community planning, resource allocation, or any context where public sentiment plays a role. For example, when evaluating attitudes toward the operation of a shop in a village, it is important to capture not only supportive and oppositional opinions but also those who remain undecided. This additional layer of information enhances the precision of decision-making processes. A practical transformation involves converting fuzzy inference systems (FIS) built with fuzzy sets into intuitionistic fuzzy inference systems (IFIS) using this approach. Along with other necessary components like inference mechanisms and intuitionistic fuzzy numbers, this transformation strengthens the system's ability to model uncertainty more effectively. The practical utility of this transformation can be seen across a variety of applications, such as supply chain management, financial forecasting, and environmental planning, where decisions are often impacted by varying degrees of uncertainty and hesitancy in expert judgments.

To further illustrate this, consider the following example:

If the experts A, B and C provide data as $(0.0,\ 0.1,\ 0.2,\ 0.3)$ with 0% hesitancy, $(0.2,\ 0.3,\ 0.4,\ 0.5)$ with 10% hesitancy and $(0.6,\ 0.7,\ 0.8,\ 0.9)$ with 20% hesitancy respectively then the intuitionistic fuzzy numbers with α level of hesitancy are addressed in Table 2. This example demonstrates how intuitionistic fuzzy numbers with α-level hesitancy provide a more accurate reflection of the experts' opinions, incorporating their hesitancy into the decision-making process. For instance, expert A, with no hesitancy, offers direct data that matches with the fuzzy representation. In contrast, expert B, with 10% hesitancy, yields an intuitionistic fuzzy number that shifts slightly to account for their uncertainty. Similarly, expert C, with 20% hesitancy, provides a more significantly adjusted intuitionistic fuzzy number. Hence, even though fuzzy inference systems (FIS) can be constructed using fuzzy sets with zero - hesitancy, this transformation allows for the development of intuitionistic fuzzy inference systems (IFIS), which further incorporate mechanisms for inference and intuitionistic fuzzy numbers. This transformation is invaluable in practice because it handles the inherent hesitancy present in human judgments more effectively. As shown in the table, the transformation allows experts' data to be interpreted within a more flexible framework that adjusts according to their hesitancy levels.

TABLE 2. Transition from Trapezoidal Fuzzy to Intuitionistic Fuzzy Numbers with α - Hesitancy

Experts	**Data**	α ($h\% = \frac{\mathsf{h}}{100}$)	**Trapezoidal Intuitionistic Fuzzy Number**
A	(0.0, 0.1, 0.2, 0.3)	0.00	[(0.0, 0.1, 0.2, 0.3), (0.0, 0.1, 0.2, 0.3)]
B	(0.2, 0.3, 0.4, 0.5)	0.10	[(0.2, 0.3, 0.4, 0.5), (0.17, 0.27, 0.45, 0.56)]
C	(0.4, 0.6, 0.8, 1.0),	0.20	[(0.4, 0.6, 0.8, 1.0), (0.33, 0.54, 0.83, 1.00)]

6|Trapezoidal Intuitionistic Fuzzy Inference Mechanisms

In this section, we discuss how the Multi-Input Single-Output (MISO) system operates when both the premises and the conclusion are represented as trapezoidal intuitionistic fuzzy numbers (IFNs). The implication rule is defined as follows:

If x is A and y is B, then w is C.

The input and output trapezoidal intuitionistic fuzzy numbers are represented as (A', B') and C' respectively. The trapezoidal intuitionistic fuzzy numbers for the premises and conclusion are defined as follows:

$$A = [(w_1, a, b, w_2), (u_1, c, d, u_2)], \quad B = [(z_1, e, f, z_2), (t_1, g, h, t_2)], \quad C = [(g_1, p, q, g_2), (h_1, r, s, h_2)]$$

For the inputs, the trapezoidal intuitionistic fuzzy numbers are:

$$A' = [(w_1', a', b', w_2'), (u_1', c', d', u_2')], \quad B' = [(z_1', e', f', z_2'), (t_1', g', h', t_2')]$$

Here, A, B, and C represent the trapezoidal intuitionistic fuzzy numbers used in the premises and conclusion of the system, while A' and B' are the inputs in trapezoidal form. The output is represented by the trapezoidal intuitionistic fuzzy number C'. By using theorem , we obtained

$$\begin{aligned} \mu_{\mathsf{C}'}(w) &= (\mu_{\mathsf{A}}(u), \mu_{\mathsf{B}}(v)) \circ \mu_{\mathsf{I}}(u, v, w) \\ &= (\mu_{\mathsf{A}}(u) \circ \mu_{\mathsf{I}'}(u, w)) \wedge (\mu_{\mathsf{B}}(v) \circ \mu_{\mathsf{I}''}(V, w)) \\ &= \mu_{\mathsf{C}_1'}(w) \wedge \mu_{\mathsf{C}_2'}(w). \end{aligned}$$

To determine $\mu_{\mathsf{C}_1'}$, let's employ following the membership function of the trapezoidal intuitionistic fuzzy numbers $\mu_{\mathsf{A}} = (w_1, a, b, w_2)$, $\mu_{\mathsf{C}} = (g_1, p, q, g_2)$ and input $\mu_{\mathsf{A}'} = (w_1', a', b', w_2')$, which are described by definition .

Case μ_1. $[a, b] \cap [a', b'] \neq \emptyset$, If $[a, b] \cap [a', b'] \neq \emptyset$ then $\mu_{\mathsf{s}} = 1$.

Case μ_2. $[a, b] \cap [a', b'] = \emptyset$, If $[a, b] \cap [a', b'] = \emptyset$ and $a' \leq b' < a \leq b$ with $w_1 < w_2'$ then $H = \{x \in [w_1, w_2'] \mid \mu_{\mathsf{A}}(x) = \mu_{\mathsf{A}'}(x)\} = H_1 \cup H_2$ where $H_1 = \{x \in [w_1, a) \mid \frac{\mathsf{x} - \mathsf{w}_1}{\mathsf{a} - \mathsf{w}_1} = \frac{\mathsf{w}_2' - \mathsf{x}}{\mathsf{w}_2' - \mathsf{b}'}\}$, $H_2 = \{x \in (b\ w'_2] \mid \frac{\mathsf{w}_2 - \mathsf{x}}{\mathsf{w}_2 - \mathsf{b}} = \frac{\mathsf{w}_2' - \mathsf{x}}{\mathsf{w}_2' - \mathsf{b}'}\}$. Since subcase $(2(i))$ in subsection 3.1.1, we can find out x as $\frac{\mathsf{a}\mathsf{w}_2' - \mathsf{w}_1 \mathsf{b}'}{(\mathsf{a} - \mathsf{b}') - (\mathsf{w}_1 - \mathsf{w}_2')}$ such that $\mu_{\mathsf{A}}(\frac{\mathsf{a}\mathsf{w}_2' - \mathsf{w}_1 \mathsf{b}'}{(\mathsf{a} - \mathsf{b}') - (\mathsf{w}_1 - \mathsf{w}_2')}) = \mu_{\mathsf{s}}$. If $[a, b] \cap [a', b'] = \emptyset$ and $a \leq b < a' \leq b'$ with $w_1' < w_2$ then $H = \{x \in [w_1', w_2] \mid \mu_{\mathsf{A}}(x) = \mu_{\mathsf{A}'}(x)\} = H_1 \cup H_2$ where $H_1 = \{x \in (b\ w_2] \mid \frac{\mathsf{w}_2 - \mathsf{x}}{\mathsf{w}_2 - \mathsf{b}} = \frac{\mathsf{x} - \mathsf{w}_1'}{\mathsf{a}' - \mathsf{w}_1'}\}$, $H_2 = \{x \in [w_1', a) \mid \frac{\mathsf{x} - \mathsf{w}_1}{\mathsf{a} - \mathsf{w}_1} = \frac{\mathsf{x} - \mathsf{w}_1'}{\mathsf{a}' - \mathsf{w}_1'}\}$. Since subcase $(2(ii))$ in subsection 3.1.1, we can find out x as $\frac{\mathsf{a}'\mathsf{w}_2 - \mathsf{w}_1' \mathsf{b}}{(\mathsf{a}' - \mathsf{b}) - (\mathsf{w}_1' - \mathsf{w}_2)}$ such that $\mu_{\mathsf{A}}(\frac{\mathsf{a}'\mathsf{w}_2 - \mathsf{w}_1' \mathsf{b}}{(\mathsf{a}' - \mathsf{b}) - (\mathsf{w}_1' - \mathsf{w}_2)}) = \mu_{\mathsf{s}}$. Hence,

$$\mu_{\mathsf{C}_1'}(w) = \begin{cases} \frac{\mathsf{w} - \mathsf{g}_1}{\mathsf{p} - \mathsf{g}_1} & \text{if } g_1 \leq w \leq x_1^* \\ \mu_{\mathsf{s}} & \text{if } x_1^* \leq w \leq y_1^* \\ \frac{\mathsf{g}_2 - \mathsf{w}}{\mathsf{g}_2 - \mathsf{q}} & \text{if } y_1^* \leq w \leq g_2 \\ 1, & \text{otherwise} \end{cases}$$

where $x_1^* = \mu_{\mathsf{s}} \cdot p + (1 - \mu_{\mathsf{s}}) \cdot g_1,\ y_1^* = \mu_{\mathsf{s}} \cdot q + (1 - \mu_{\mathsf{s}}) \cdot g_2$ and denoted as

$$\mu_{\mathsf{C}_1'} = [(g_1, x_1^*, y_1^*, g_2) : \mu_{\mathsf{s}}]. \tag{19}$$

Similarly, if we take into consideration the membership function of trapezoidal intuitionistic fuzzy numbers $\mu_B = (z_1, e, f, z_2)$, $\mu_C = (g_1, p, q, g_2)$ and input $\mu_{B'} = (z_1', e', f', z_2')$, we can obtain $\mu_{C_2'} = [(g_1, x_2^*, y_2^*, g_2) : \mu_s']$. By theorem ,

$$\mu_{C'}(w) = \mu_{C_1'}(w) \wedge \mu_{C_2'}(w)$$

where $\mu_{C'}$ is a membership function of the final output C' when $\mu_{A'}$ and $\mu_{B'}$ are the inputs. Now

$$\begin{aligned}\nu_{C'}(w) &= (\nu_A(u), \nu_B(v)) \circ \nu_I(u, v, w)\\ &= (\nu_A(u) \circ \nu_{I'}(u, w)) \vee (\nu_B(v) \circ \nu_{I''}(v, w))\\ &= \nu_{C_1'}(w) \vee \nu_{C_2'}(w).\end{aligned}$$

To determine $\nu_{C_1'}$, let's employ following the non-membership function of the trapezoidal intuitionistic fuzzy numbers $\nu_A = (u_1, c, d, u_2)$, $\nu_C = (h_1, r, s, h_2)$ and input $\nu_{A'} = (u_1', c', d', u_2')$, which are described by definition .

Case ν_1. $[c, d] \cap [c', d'] \neq \emptyset$, If $[c, d] \cap [c', d'] \neq \emptyset$ then $\nu_s = 0$.

Case ν_2. $[c, d] \cap [c', d'] = \emptyset$, If $[c, d] \cap [c', d'] = \emptyset$ and $c' \leq d' < c \leq d$ with $u_1 < u_2'$ then $K = \{y \in [u_1, u_2'] \mid \nu_A(y) = \nu_{A'}(y)\}$ $= K_1 \cup K_2$ where $K_1 = \{y \in [u_1, c) \mid \frac{c-y}{c-u_1} = \frac{y-d'}{u_2'-d'}\}$, $K_2 = \{y \in (d\, u'_2] \mid \frac{y-d}{u_2-d} = \frac{y-d'}{u_2'-d'}\}$. Since subcase $(2(i))$ in subsection 3.1.2, we can find out y as $\frac{cu_2'-d'u_1}{(c-d')-(u_1-u_2')}$ such that $\nu_A(\frac{cu_2'-d'u_1}{(c-d')-(u_1-u_2')}) = \nu_s$. If $[c, d] \cap [c', d'] = \emptyset$ and $c' \leq d' < c \leq d$ with $u_1 < u_2'$ then $K = \{y \in [u_1, u_2'] \mid \nu_A(y) = \nu_{A'}(y)\} = K_1 \cup K_2$ where $K_1 = \{y \in [u_1, c) \mid \frac{c-y}{c-u_1} = \frac{y-d'}{u_2'-d'}\}$, $K_2 = \{y \in (d\, u'_2] \mid \frac{y-d}{u_2-d} = \frac{y-d'}{u_2'-d'}\}$. Since subcase $(2(i))$ in subsection 3.1.2, we can find out y as $\frac{cu_2'-d'u_1}{(c-d')-(u_1-u_2')}$ such that $\nu_A(\frac{cu_2'-d'u_1}{(c-d')-(u_1-u_2')}) = \nu_s$. If $[c, d] \cap [c', d'] = \emptyset$ and $c \leq d < c' \leq d'$ with $u_1' < u_2$ then $K = \{y \in [u_1', u_2] \mid \nu_A(y) = \nu_{A'}(y)\}$ $= K_1 \cup K_2$ where $K_1 = \{y \in (d\; u_2] \mid \frac{y-d}{u_2-d} = \frac{c'-y}{c'-u_1'}\}$, $K_2 = \{y \in [u_1', c) \mid \frac{c-y}{c-u_1} = \frac{c'-y}{c'-u_1'}\}$. Since subcase $(2(ii))$ in subsection 3.1.2, we can find out y as $\frac{c'u_2-du_1'}{(c'-d)-(u_1'-u_2)}$ such that $\nu_A(\frac{c'u_2-du_1'}{(c'-d)-(u_1'-u_2)}) = \nu_s$. Hence,

$$\nu_{C_1'}(w) = \begin{cases} \frac{r-w}{r-h_1} & \text{if } h_1 \leq w \leq m_1^* \\ \nu_s & \text{if } m_1^* \leq w \leq n_1^* \\ \frac{w-s}{h_2-s} & \text{if } n_1^* \leq w \leq h_2 \\ 1, & \text{otherwise} \end{cases}$$

where $m_1^* = \nu_s \cdot h_1 + (1-\nu_s) \cdot r,\ n_1^* = \nu_s \cdot h_2 + (1-\nu_s) \cdot s$ and denoted as

$$\nu_{C_1'} = [(h_1, m_1^*, n_1^*, h_2) : \nu_s]. \tag{20}$$

Similarly, if we take into account the non-membership function of trapezoidal intuitionistic fuzzy numbers $\nu_B = (t_1, g, h, t_2)$, $\nu_C = (h_1, r, s, h_2)$ and input $\nu_{B'} = (t_1', g', h', t_2')$, we can obtain $\nu_{C_2'} = [(h_1, m_2^*, n_2^*, h_2) : \nu_s']$. By theorem ,

$$\nu_{C'}(w) = \nu_{C_1'}(w) \vee \nu_{C_2'}(w)$$

where $\nu_{C'}$ is a non-membership function of the final output C' when $\nu_{A'}$ and $\nu_{B'}$ are the inputs. Therefore, the final out put is $C' = (\mu_C', \nu_C')$ when A' and B' are inputs.

7|Numerical illustration

In this section, we apply the Intuitionistic Fuzzy Inference System (IFIS) to assess the risk of COVID-19 based on critical technical parameters. The framework and parameters are adopted from [16], with the assumption that the dataset includes a 10% lack of confidence. The input parameters considered are Body Temperature (**BT**), Body Immunity Level (**BI**), and Efficacy of the Vaccine (**EV**), while the output parameter is the Change of Occurrence of COVID-19 (**COC**). According to current COVID-19 literature, each parameter is classified into one of three levels: Low (**L**), Moderate (**M**), or Peak (**P**), which are represented as intuitionistic fuzzy numbers. The IFIS evaluates all possible combinations of the three input parameters, resulting in 27 distinct intuitionistic fuzzy implications, which are implemented using the Mamdani IFI approach, as described in Section 1. The fuzzy numbers for L, M, and P are referenced from [16] (Table I) and normalized to the range $[0, 1]$. These normalized values are then transformed into intuitionistic fuzzy numbers with a 10% level of hesitancy using the proposed method outlined in Section 5 for converting trapezoidal fuzzy numbers into

TABLE 3. Input and output specifications represented by trapezoidal intuitionistic numbers

Input Specifications	**L**	**M**	**P**
Body temperature	[(0.00, 0.15, 0.23, 0.31), (0.00, 0.12, 0.30, 0.37)]	[(0.15, 0.31, 0.46, 0.62), (0.12, 0.28, 0.51, 0.65)]	[(0.46, 0.62, 0.77, 1.00), (0.43, 0.59, 0.79, 1.00)]
Body immunity level	[(0.00, 0.10, 0.20, 0.30), (0.00, 0.08, 0.27, 0.36)]	[(0.20, 0.30, 0.40, 0.50), (0.17, 0.27, 0.45, 0.54)]	[(0.40, 0.60, 0.80, 1.00), (0.36, 0.57, 0.82, 1.00)]
Efficacy of vaccine	[(0.00, 0.10, 0.20, 0.40), (0.00, 0.08, 0.27, 0.45)]	[(0.30, 0.50, 0.60, 0.70), (0.27, 0.47, 0.64, 0.73)]	[(0.60, 0.70, 0.80, 1.00), (0.57, 0.68, 0.82, 1.00)]
Chance of occurrence of COVID-19	[(0.00, 0.10, 0.20, 0.30), (0.00, 0.08, 0.27, 0.36)]	[(0.20, 0.40, 0.50, 0.60), (0.17, 0.36, 0.54, 0.64)]	[(0.50, 0.70, 0.80, 1.00), (0.47, 0.68, 0.82, 1.00)]

intuitionistic fuzzy numbers. The transformed values are presented in Table 3. This approach provides a comprehensive mechanism for assessing COVID-19 risk through the integration of multiple input factors into the IFIS framework.

The normalization was functioning as explained below. To do this, take the data for BT in [16, Table I] and performing the method

$$\left[\frac{X - X_{\min}}{X_{\max} - X_{\min}}\right]$$

where $X_{\min} = \min\{X \mid X \in [92, 105]\} = 92$ and $X_{\max} = \max\{X \mid X \in [92, 105]\} = 105$ to obtain the normalised value. The underlying implications administering the IFIS are expressed in Tabel 4. The forthcoming inputs are IFNs with a (10%) 0.1 $-$ level of hesitancy for IFIS, which are set up by a detailed study in [1], [15, Table 2], [16], [21], [23], [25], [26], [29], [42], [43], [44], [45], [46], [48], [50]. There are

$$\begin{aligned} BT' &= [(0.52,\ 0.57,\ 0.67,\ 0.72), (0.49,\ 0.54,\ 0.70,\ 0.75)], \\ BI' &= [(0.70,\ 0.75,\ 0.85,\ 0.90), (0.68,\ 0.73,\ 0.86,\ 0.91)] \text{ and} \\ EV' &= [(0.69,\ 0.74,\ 0.84,\ 0.89), (0.66,\ 0.72,\ 0.85,\ 0.90)] \end{aligned}$$

then by applying theorem , the result COC' is generated as follows,

$$COC'(w) = \big(\max\{\mu_{\mathsf{COC}\,{}_1'}(w), \mu_{\mathsf{COC}\,{}_2'}(w) \cdots \mu_{\mathsf{COC}\,{}_{27}'}(w)\}, \min\{\nu_{\mathsf{COC}\,{}_1'}(w), \nu_{\mathsf{COC}\,{}_2'}(w) \cdots \nu_{\mathsf{COC}\,{}_{27}'}(w)\}\big). \tag{21}$$

Our goal is to determine the membership function μ_{s} and the non-membership function ν_{s} for a Multiple-Input Single-Output (MISO) system. Using equations 10 and 11, the weights of the premises for rule j are computed as follows:

$$(\mu_{\mathsf{s}_\mathsf{j}}, \nu_{\mathsf{s}_\mathsf{j}}) = (\mu_{\mathsf{s}_{\mathsf{BT}\ \mathsf{j}}} \wedge \mu_{\mathsf{s}_{\mathsf{BI}\ \mathsf{j}}} \wedge \mu_{\mathsf{s}_{\mathsf{EV}\ \mathsf{j}}}, \nu_{\mathsf{s}_{\mathsf{BT}\ \mathsf{j}}} \vee \nu_{\mathsf{s}_{\mathsf{BI}\ \mathsf{j}}} \vee \nu_{\mathsf{s}_{\mathsf{EV}\ \mathsf{j}}}) \tag{22}$$

where $(\mu_{\mathsf{s}_{\mathsf{BT}\ \mathsf{j}}}, \nu_{\mathsf{s}_{\mathsf{BT}\ \mathsf{j}}}), (\mu_{\mathsf{s}_{\mathsf{BI}\ \mathsf{j}}}, \nu_{\mathsf{s}_{\mathsf{BI}\ \mathsf{j}}}), (\mu_{\mathsf{s}_{\mathsf{EV}\ \mathsf{j}}}, \nu_{\mathsf{s}_{\mathsf{EV}\ \mathsf{j}}})$ represent the weights of premises for the implications $BT_\mathsf{j} \to COC_\mathsf{j}$, $BI_\mathsf{j} \to COC_\mathsf{j}$, and $EV_\mathsf{j} \to COC_\mathsf{j}$, respectively.

Calculation for $j = 18$

Considering $j = 18$, the rule is: **If** (BT is M), (BI is P), and (EV is P), **then** (COC is L). The weights of the premises for $j = 18$ are:

$$(\mu_{\mathsf{s}_{18}}, \nu_{\mathsf{s}_{18}}) = (\mu_{\mathsf{s}_{\mathsf{BT}\ 18}} \wedge \mu_{\mathsf{s}_{\mathsf{BI}\ 18}} \wedge \mu_{\mathsf{s}_{\mathsf{EV}\ 18}}, \nu_{\mathsf{s}_{\mathsf{BT}\ 18}} \vee \nu_{\mathsf{s}_{\mathsf{BI}\ 18}} \vee \nu_{\mathsf{s}_{\mathsf{EV}\ 18}}).$$

To compute $(\mu_{\mathsf{s}_{\mathsf{BT}\ 18}}, \nu_{\mathsf{s}_{\mathsf{BT}\ 18}}), (\mu_{\mathsf{s}_{\mathsf{BI}\ 18}}, \nu_{\mathsf{s}_{\mathsf{BI}\ 18}}), (\mu_{\mathsf{s}_{\mathsf{EV}\ 18}}, \nu_{\mathsf{s}_{\mathsf{EV}\ 18}})$, we refer to section 6 to determine the specific case for the provided data.

Calculation of $(\mu_{\mathsf{s}_{\mathsf{BT}\ 18}}, \nu_{\mathsf{s}_{\mathsf{BT}\ 18}})$:

- Given data: $[0.31, 0.46] \cap [0.57, 0.67] = \emptyset$ and $[0.28, 0.51] \cap [0.54, 0.70] = \emptyset$.
- This corresponds to case μ_2 and case ν_2 in section 6.

TABLE 4. Intuitionistic IF-THEN Implications for IFIS

	BT	BI	EV	COC
01	L	L	L	M
02	L	L	M	L
03	L	L	P	L
04	L	M	L	L
05	L	M	M	L
06	L	M	P	L
07	L	P	L	L
08	L	P	M	L
09	L	P	P	L
10	M	L	L	P
11	M	L	M	L
12	M	L	P	L
13	M	M	L	M
14	M	M	M	L
15	M	M	P	L
16	M	P	L	L
17	M	P	M	L
18	M	P	P	L
19	P	L	L	M
20	P	L	M	M
21	P	L	P	L
22	P	M	L	L
23	P	M	M	M
24	P	M	P	L
25	P	P	L	M
26	P	P	M	L
27	P	P	P	L

- Calculating the weights: $w_{bt} = 0.544$ and $w'_{bt} = 0.532$, we get:

$$\mu_{s_{BT_{18}}} = \mu_{BT}(0.544) = 0.48, \quad \nu_{s_{BT_{18}}} = \nu_{BT}(0.532) = 0.16.$$

Thus, $(\mu_{s_{BT_{18}}}, \nu_{s_{BT_{18}}}) = (0.48, 0.16)$.

Calculation of $(\mu_{s_{BI_{18}}}, \nu_{s_{BI_{18}}})$:

- Given data: $[0.60, 0.80] \cap [0.75, 0.85] = [0.75, 0.80] \neq \emptyset$ and $[0.57, 0.82] \cap [0.73, 0.86] = [0.73, 0.82] \neq \emptyset$.
- This corresponds to case μ_1 and case ν_1 in section 6.
- Calculating the weights: $w_{bi} = 0.75$ and $w'_{bi} = 0.73$, we get:

$$\mu_{s_{BI_{18}}} = \mu_{BI}(0.75) = 1.00, \quad \nu_{s_{BI_{18}}} = \nu_{BI}(0.73) = 0.00.$$

Thus, $(\mu_{s_{BI_{18}}}, \nu_{s_{BI_{18}}}) = (1.00, 0.00)$.

Table 5. Weights of Premises and Outputs COC_j' for Each Implication.

I (j)	$\mu_{S_{BT\ j}}$	$\nu_{S_{BT\ j}}$	$\mu_{S_{BI\ j}}$	$\nu_{S_{BI\ j}}$	$\mu_{S_{EV\ j}}$	$\nu_{S_{EV\ j}}$	μ_S	ν_S	$\mu_{COC\ j}'$	$\nu_{COC\ j}'$
1	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
2	0.00	1.00	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
3	0.00	1.00	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
4	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
5	0.00	1.00	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
6	0.00	1.00	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
7	0.00	1.00	1.00	0.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
8	0.00	1.00	1.00	0.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
9	0.00	1.00	1.00	0.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
10	0.48	0.16	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
11	0.48	0.16	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
12	0.48	0.16	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
13	0.48	0.16	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
14	0.48	0.16	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
15	0.48	0.16	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
16	0.48	0.16	1.00	0.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
17	0.48	0.16	1.00	0.00	0.067	0.53	0.067	0.53	(0.0, 0.0067, 0.293, 0.3 : 0.067)	(0.0, 0.0376, 0.3177, 0.36 : 0.53)
18	0.48	0.16	1.00	0.00	1.00	0.00	0.48	0.16	(0.0, 0.048, 0.252, 0.3 : 0.48)	(0.0, 0.0672, 0.2844, 0.36 : 0.16)
19	1.00	0.00	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
20	1.00	0.00	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
21	1.00	0.00	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
22	1.00	0.00	0.00	1.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
23	1.00	0.00	0.00	1.00	0.067	0.53	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
24	1.00	0.00	0.00	1.00	1.00	0.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
25	1.00	0.00	1.00	0.00	0.00	1.00	0.00	1.00	(0.0, 0.0, 0.0, 0.0 : 0.0)	(1.0, 1.0, 1.0, 1.0 : 1.0)
26	1.00	0.00	1.00	0.00	0.067	0.53	0.067	0.53	(0.0, 0.0067, 0.293, 0.3 : 0.067)	(0.0, 0.0376, 0.3177, 0.36 : 0.53)
27	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00	(0.0, 0.1, 0.2, 0.3 : 1.00)	(0.0, 0.08, 0.27, 0.36 : 0.00)

Calculation of $(\mu_{S_{EV\ 18}}, \nu_{S_{EV\ 18}})$:

- Given data: $[0.70, 0.80] \cap [0.74, 0.84] = [0.74, 0.80] \neq \emptyset$ and $[0.68, 0.82] \cap [0.72, 0.85] = [0.72, 0.80] \neq \emptyset$.

- This corresponds to case μ_1 and case ν_1 in section 6.
- Calculating the weights: $w_{ev} = 0.74$ and $w'_{ev} = 0.72$, we get:

$$\mu_{s_{EV_{18}}} = \mu_{EV}(0.74) = 1.00, \quad \nu_{s_{EV_{18}}} = \nu_{EV}(0.72) = 0.00.$$

Thus, $(\mu_{s_{EV_{18}}}, \nu_{s_{EV_{18}}}) = (1.00, 0.00)$.

Final Weight of Premises

By substituting the computed values into equation (22), we obtain:

$$(\mu_s, \nu_s) = (0.48 \wedge 1.00 \wedge 1.00, 0.16 \vee 0.00 \vee 0.00) = (0.48, 0.16).$$

Output Calculation

Using the inverse membership and non-membership functions, we can calculate:

$$\mu^{-1}_{COC_{18}}[0.48, 1] = [0.048, 0.252], \quad \nu^{-1}_{COC_{18}}[0, 0.16] = [0.0672, 0.2844].$$

Thus, the output for COC'_{18} is:

$$COC'_{18} = ([0.0, 0.048, 0.252, 0.3 : 0.48], [0.0, 0.0672, 0.2844, 0.36 : 0.16]).$$

For $j = 1, 2, \ldots, 17, 19, \ldots, 27$, similar calculations apply.

Final Output for IFIS

The required output for the Intuitionistic Fuzzy Inference System (IFIS) is:

$$COC' = [(0.0, 0.1, 0.2, 0.3 : 1), (0.0, 0.08, 0.27, 0.36 : 0)].$$

By applying the intuitionistic defuzzification method from equation 15, the crisp output is 0.1661 when $\Theta = 0.5$.

8|Comparative Analysis

In this section, we present a detailed comparative analysis of the proposed Multi-Input Single-Output (MISO) Intuitionistic Fuzzy Inference System (IFIS) approach. The analysis is carried out by applying our proposed method of IFIS to method of FIS with trapezoidal fuzzy numbers at different levels of hesitancy, as described in Section 5, and then extending the discussion in Section 6. The IFIS method is further illustrated in Section 7 by considering an example problem from [16]. In this example, the dataset is assumed to be imprecise and incomplete, a result of the inherent hesitancy assumed in the data due to lack of confidence. A comparative analysis is presented in Table 6, where we apply the same Fuzzy Inference System (FIS) described in [16], but with varying degrees of hesitancy in the dataset.

8.1 |Impact of Hesitancy Levels on the Crisp Output

We observe that as the level of hesitancy in the input data increases, the crisp output generated by the IFIS method significantly changes. This is particularly evident when comparing the outputs obtained by IFIS with those generated by the standard FIS approach, which does not account for hesitancy. When the level of hesitancy is set to zero, i.e., when $\alpha = 0$, we observe that the crisp output value for given input values using IFIS is 0.1500, which coincides with the crisp output generated by FIS for same input values. This observation confirms that when there is no hesitancy in the data, IFIS behaves as FIS. As the level of hesitancy increases slightly, from $\alpha = 0.0$ to $\alpha = 0.10$, the crisp output value changes from 0.1500 to 0.1661. This small shift indicates that the inclusion of hesitancy affects the crisp output, resulting in a more accurate reflection of the underlying uncertainty in the data. As hesitancy level rises further to $\alpha = 0.20$, the crisp output increases to 0.1981. This difference highlights the impact of growing uncertainty in the input data. The output value differs from both the earlier values of 0.1500 (no hesitancy) and 0.1661 (10% hesitancy), demonstrating that higher levels of hesitancy result in progressively greater deviations from the original crisp output.

8.2 |IFIS vs. FIS: A Comparative Perspective

The key observation from this analysis is that the IFIS method provides more accurate and reliable outputs in scenarios where the dataset is incomplete or imprecise due to the presence of hesitancy. When comparing IFIS outputs to those from the traditional FIS approach, the outputs from both methods are identical at $\alpha = 0.0$ as expected, since no hesitancy is considered. In this case, IFIS is essentially reduced to FIS, where the intuitionistic fuzzification collapses to standard fuzzification. As soon as hesitancy is introduced ($\alpha > 0.0$), the output of the IFIS begins to diverge from that of FIS. This divergence reflects the additional information captured by IFIS concerning the degree of uncertainty or lack of knowledge in the dataset. By incorporating hesitancy into the analysis, the IFIS approach provides a more nuanced and accurate output compared to the traditional FIS in which it is assumed that the input data is always precise with 100% confident may be incorrect in human based evaluations. Therefore, in real-world applications where imprecision and incomplete knowledge are common, IFIS proves to be a superior alternative. Table 6 summarizes the results of the comparative analysis. It displays the crisp output values generated by IFIS for different levels of hesitancy, demonstrating the impact of hesitancy on the overall system output. The proposed MISO IFIS method performs more robustly in environments

Table 6. Significance of Proposed MISO-IFIS Method

Level of Hesitancy $\alpha = (h\% = \frac{\mathsf{h}}{100})$	**Intuitionistic Fuzzy Out-put**	**Structure of the System**	**Crisp Out-put**
0.0 (0%)	$[(0.0,\ 0.1,\ 0.2,\ 0.3:\ 1),(0.0,\ 0.1,\ 0.2,\ 0.3:\ 0.0)]$	FIS	0.1500
0.1 (10%)	$[(0.0,\ 0.1,\ 0.2,\ 0.3:\ 1),(0.0,\ 0.08,\ 0.27,\ 0.36:0.0)]$	MISO - IFIS	0.1661
0.2 (20%)	$[(0.0,\ 0.1,\ 0.2,\ 0.3:\ 1),(0.0,\ 0.05,\ 0.38,\ 0.46:0.0)]$	MISO - IFIS	0.1981

where data is subject to uncertainty and hesitancy. As the level of hesitancy increases, the crisp output generated by IFIS adapts accordingly, offering a more accurate and reliable solution compared to the traditional FIS approach. This supports our assertion that FIS is, in fact, a specific case of IFIS, where the level of hesitancy (α) is set to zero.

8.2 |Comparative Analysis of MISO IFIS Approach with Existing SISO IFIS Methods

In this section, we present a comprehensive comparative analysis of the proposed Multi-Input Single-Output (MISO) Intuitionistic Fuzzy Inference System (IFIS) approach against existing Single-Input Single-Output (SISO) IFIS methods. This analysis aims to highlight the strengths of our MISO-IFIS model, especially in the context of handling data uncertainty and hesitancy levels.

8.2.1 |Limitations of Existing SISO IFIS Approaches

Previous works on Intuitionistic Fuzzy Inference Systems (IFIS) such as [32],[22], [51], [54], [2], [19] and [41], have primarily relied on intuitionistic fuzzy numbers (IFNs) for data representation. While these approaches are mathematically rigorous, they introduce specific limitations that hinder their applicability in various domains, including the complexity posed for non-experts, as the representation of data as intuitionistic fuzzy numbers can be mathematically intricate. This complexity creates challenges for technical experts who may not be well-versed in the theory of intuitionistic fuzzy sets, making the application of such systems less feasible in practical contexts. Furthermore, existing methods have predominantly been designed for Single-Input Single-Output (SISO) problems, limiting their flexibility to handle Multi-Input Single-Output (MISO) scenarios which are more commonly encountered in real-world applications, thereby highlighting the need for an extended framework. Additionally, current IFIS approaches do not adequately address issues related to varying degrees of hesitancy, denoted by α, which is particularly critical in scenarios characterized by uncertainty and indecision in input data, ultimately reducing the robustness of the system. In light of these challenges, we propose a MISO-IFIS model that extends the SISO framework which is effectively managing multiple inputs while addressing the inherent limitations associated with data uncertainty and hesitancy.

8.2.1 | Comparative Analysis of MISO-IFIS with Existing SISO-IFIS Methods

To illustrate the effectiveness of our proposed MISO-IFIS, we first examine a typical SISO-IFIS approach adapted from [32]. Consider two intuitionistic fuzzy sets $Ai(x) = (\mu Ai(x), \nu Ai(x))$ and $Bi(y) = (\mu Bi(y), \nu Bi(y))$ that represent the fuzzy rules: If x is $Ai(x)$ then y is $Bi(y)$. We utilize the following triangular intuitionistic fuzzy numbers (TIFNs): $A_1 = ((1,2,3); 0.90, 0.05), A_2 = ((2,3,4); 0.80, 0.15)$, $B_1 = ((2,4,6); 0.95, 0.00)$ and $B_2 = ((4,6,8); 0.90, 0.10)$. Given a crisp input value $x_0 = 2.5$ using the membership and non-membership functions of a triangular intuitionistic fuzzy number$A = ((a,b,c); \beta, \gamma)$ as:

$$\mu A(x) = \begin{cases} \beta\left(\dfrac{x-a}{b-a}\right) & \text{if} \quad a \le x \le b \\ \beta\left(\dfrac{c-x}{c-b}\right) & \text{if} \quad b \le x \le c \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu A(x) = \begin{cases} \dfrac{b-x+\gamma(x-a)}{b-a} & \text{if } a \le x \le b \\ \dfrac{x-b+\gamma(c-x)}{c-b} & \text{if } b \le x \le c \\ 1, & \text{Otherwise} \end{cases}.$$

Using the intuitionistic fuzzy reasoning approach, the output fuzzy set $B'(y) = (\mu B'(y), \nu B'(y))$ is determined as follows:

$$\mu B(y) = \begin{cases} \dfrac{0.95}{2}(y-2) & \text{if} \quad 2 \le y < 3.05 \\ 0.45 & \text{if} \quad 3.05 \le y < 5.05 \\ \dfrac{0.95}{2}(6-y) & \text{if} \quad 5.05 \le y < 5.16 \\ 0.40 & \text{if} \quad 5.16 \le y < 7.11 \\ \dfrac{0.90}{2}(8-y) & \text{if} \quad 7.11 \le y << 8 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu B(y) = \begin{cases} \dfrac{1}{2}(4-y) & \text{if} \quad 2 \le y < 2.95 \\ 0.525 & \text{if} \quad 2.95 \le y < 5.05 \\ \dfrac{1}{2}(y-4) & \text{if} \quad 5.05 \le y < 5.15 \\ 0.575 & \text{if} \quad 5.15 \le y < 7.01 \\ \dfrac{0.95}{2}(y-6)+0.10 & \text{if} \quad 7.06 \le y << 8 \\ 1, & \text{otherwise} \end{cases}$$

After applying an intuitionistic defuzzification operator, the crisp output value for the SISO system is calculated as $y^{0'} = 4.97$.

Our proposed MISO-IFIS extends the SISO framework to handle multiple inputs and outputs. In this case, the fuzzy reasoning process incorporates multiple premises, and the resulting intuitionistic fuzzy output B' is a combination of individual fuzzy implications.

TABLE 7. Weights of Premises and Outputs B'_j for Each Implication.

$I\ (j)$	μ_s	ν_s	$\mu_{B'_j}$	$\nu_{B'_j}$
1	0.45	0.525	(2.00, 2.95, 5.05, 6.00 : 0.45)	(2.00, 2.95, 5.05, 6.00 : 0.525)
2	0.40	0.575	(4.00, 4.90, 7.11, 8.00 : 0.40)	(4.00, 4.95, 7.06, 8.00 : 0.575)

The intuitionistic fuzzy output of the MISO-IFIS system is computed as:

$$B'(y) = (\mu_{B'_1}(y) \vee \mu_{B'_2}(y),\ \nu_{B'_1}(y) \wedge \nu_{B'_2}(y))$$

where $\mu B'_1 = (2.00,\ 2.95,\ 5.05,\ 6.00\ :\ 0.45)$, $\mu B'_2 = (4.00,\ 4.90,\ 7.11,\ 8.00\ :\ 0.40)$, $\nu B'_1 = (2.00,\ 2.95,\ 5.05,\ 6.00\ :\ 0.525)$ and $\nu B'_2 = (4.00,\ 4.95,\ 7.06,\ 8.00\ :\ 0.575)$. The membership and non-membership functions of the resulting intuitionistic

fuzzy output are given by:

$$\mu_{\mathsf{B}'}(y) = \begin{cases} \frac{0.95}{2}(y-2) & \text{if} \quad 2 \le y < 3.05 \\ 0.45 & \text{if} \quad 3.05 \le y < 5.05 \\ \frac{0.95}{2}(6-y) & \text{if} \quad 5.05 \le y < 5.16 \\ 0.40 & \text{if} \quad 5.16 \le y < 7.11 \\ \frac{0.90}{2}(8-y) & \text{if} \quad 7.11 \le y <\le 8 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\mathsf{B}'}(y) = \begin{cases} \frac{1}{2}(4-y) & \text{if} \quad 2 \le y < 2.95 \\ 0.525 & \text{if} \quad 2.95 \le y < 5.05 \\ \frac{1}{2}(y-4) & \text{if} \quad 5.05 \le y < 5.15 \\ 0.575 & \text{if} \quad 5.15 \le y < 7.01 \\ \frac{0.95}{2}(y-6)+0.10 & \text{if} \quad 7.06 \le y <\le 8 \\ 1, & \text{otherwise} \end{cases}$$

After applying the intuitionistic defuzzification operator, the crisp output value of the MISO-IFIS is calculated to be 4.94, which is very close to the SISO system's output of 4.97. Our proposed MISO-IFIS successfully extends the existing SISO-IFIS approach by accommodating multiple inputs and producing comparable crisp outputs. Despite the added complexity of handling multiple inputs, the results remain consistent, demonstrating the robustness and effectiveness of the proposed model. Thus, our method provides a more flexible and practical solution for real-world applications where multiple inputs are essential, while maintaining the strengths of existing intuitionistic fuzzy systems. This comparative analysis confirms that our MISO-IFIS is a valid extension of the SISO framework, as previously presented by J. Li et al. (2019) in [32].

Existing inference systems mainly focus on fuzzy scenarios, specifically through traditional Fuzzy Inference Systems (FIS) and Single-Input Single-Output Intuitionistic Fuzzy Inference Systems (SISO IFIS), as shown in Table 8. However, these systems have limitations when it comes to flexibility and application to different situations. Our proposed method stands out because it can work in both fuzzy and intuitionistic fuzzy contexts, covering both SISO IFIS and Multi-Input Single-Output Intuitionistic Fuzzy Inference Systems (MISO IFIS). This ability makes our method more versatile and easier to apply in various situations. Additionally, in traditional trapezoidal FIS, dealing with hesitancy data is challenging. Our proposed method addresses this issue by providing a way to convert trapezoidal fuzzy numbers into trapezoidal intuitionistic fuzzy numbers. This transformation helps to manage hesitancy data effectively, making inference systems more adaptable in complex decision-making situations.

Table 8. Methods Comparison Table

Authors	**Existing Systems**			**Limitation of Existing Systems**
	FIS	**SISO - IFIS**	**MISO - IFIS**	
J. Cao et al. (2024) [11] R. Pabarja (2024) [40] S. Tutun et al. (2023) [49] I. E. Ahmed et al. (2023) [3] S. N. Nam et al. (2023) [36] P. A. Darwito et al. (2023) [14] V. Navale et al. (2023) [37] R. Tabbussum et al. (2021) [47] J. Amani et al. (2012) [4] E. H. Mamdani et al. (1999) [34] J. L. Castro et al. (1996) [13] J. L. Castro (1995) [12] Chuen-Chien Lee (1990) [30], [31]	**Yes**	**No**	**No**	**IFIS**
W. Wang et al. (2023) [51] O. C. Yolcu et al. (2022) [54] M. Dhyani et al. (2022) [19] H. M. Pauzi et al. (2022) [41] J. Li et al. (2019) [32] E. Egrioglu et al. (2019) [22] A. H. Aguila al. (2016) [2]	**Yes**	**Yes**	**No**	**MISO - IFIS**

9|Conclusions

This study advances the field of Intuitionistic Fuzzy Inference Systems (IFIS) by providing a detailed analysis of both Single-Input Single-Output (SISO) and Multi-Input Single-Output (MISO) frameworks. We introduced a robust mathematical methodology for intuitionistic fuzzy mechanisms, enabling more accurate comparison of inputs and implications, addressing the limitations of traditional fuzzy systems in handling incomplete or vague information. A pivotal contribution of this work is the development of a novel intuitionistic defuzzification operator, which allows for direct conversion of intuitionistic fuzzy numbers into crisp outputs in a single stage. This approach streamlines the conversion process and improves both computational efficiency and accuracy. Furthermore, we extended this to the Trapezoidal Intuitionistic Fuzzy Inference System (TIFIS), demonstrating its ability to manage trapezoidal intuitionistic fuzzy numbers while accounting for expert hesitancy. This innovation is especially valuable when working with uncertain or incomplete data, allowing for more reliable decision-making. We also introduced the process of converting trapezoidal fuzzy numbers into intuitionistic trapezoidal fuzzy numbers with an α - level of hesitation to address scenarios involving a lack of confidence in data. This approach provides a novel method for studying fuzzy inference systems (FIS) under uncertainty. In our comparative analysis, we demonstrated that the proposed MISO intuitionistic fuzzy system offers significant advantages over traditional fuzzy systems by integrating both membership and non-membership functions, allowing for more nuanced reasoning and reliable outcomes. Additionally, we applied our framework to develop a COVID-19 risk model, illustrating the practical application of TIFIS in a real-world scenario. By incorporating uncertainty and hesitancy in expert data, the model provides a more accurate risk assessment for pandemic management, highlighting the practical potential of IFIS in handling complex, uncertain environments.

Directions for Future Research

To build upon the foundations established in this research, several concrete avenues for future investigation are proposed:

(1) MIMO Systems Development: Expanding the framework to include Multi-Input Multi-Output (MIMO) intuitionistic fuzzy systems could enhance decision-making in more complex scenarios.

(2) Integration with Other Fuzzy Models: Investigating the integration of our approach with other models, such as hesitant fuzzy, Pythagorean fuzzy, and neutrosophic models, could lead to more flexible tools for managing uncertainty.

(3) Empirical Studies: Further applications of MISO intuitionistic fuzzy systems, particularly in fields like public health or financial risk assessment, will provide deeper insights into their practical effectiveness.

(4) Optimizing Defuzzification Techniques: Exploring alternative defuzzification methods and their impact on system performance will continue to enhance the robustness of intuitionistic fuzzy inference systems.

By exploring these avenues, future research can expand the applicability of IFIS, contributing to more advanced and reliable decision-making tools in uncertain environments.

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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